Introduction to wave mechanics: Dirac equation

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The quantum mechanical Dirac equation is a deterministic equation describing the evolution of spin angular momentum density (or spin density). Therefore an understanding of the classical physics description of spin density is a logical prerequisite for understanding quantum mechanics. This paper outlines how a classical theory of spin density can be represented by a first-order wave equation for Dirac bispinors. This approach offers students a conceptual bridge between classical physics and quantum mechanics.

We specifically address two common misconceptions by demonstrating that special relativity and spin angular momentum are consequences of classical wave theory. First, a wave equation is derived for infinitesimal shear waves in an elastic solid. Next, a change of variables is used to describe the waves in terms of classical spin density - the field whose curl is equal to twice the classical momentum density. The second-order wave equation is then converted to a first-order Dirac equation: first in one dimension and then generalized to three dimensions. Conceptually, the Dirac equation is much easier to understand than the Schrödinger equation for two reasons: (1) the wave function has a well-defined physical interpretation, and (2) consistency with special relativity is guaranteed by Lorentz-invariance of the wave equation. Bispinors describing transverse plane wave solutions are presented. These contain a phase factor with half of the phase of the real-valued vector wave functions. Hence spherical harmonics with odd and even angular quantum numbers (ℓ) are analogous to fermions and bosons, respectively. The classical operator for spatial reflection is equivalent to the quantum mechanical transformation between matter and antimatter. The dynamical operators of relativistic quantum mechanics are derived. Additivity of spin density in wave interactions is the basis for the Pauli exclusion principle and interaction potentials.

Keywords: classical interpretation, Dirac equation, elastic solid, parity, quantum mechanics pedagogy, spin angular momentum, spin density, teaching quantum mechanics, wave mechanics

1. INTRODUCTION

Students of physics are typically introduced to quantum mechanics via the Schrödinger equation. Although this equation can successfully describe some processes, it suffers from the fact that unlike ordinary wave equations it is not Lorentz-invariant. The Schrödinger equation also does not provide an interpretation of spin angular momentum, which is intrinsic to elementary particles. Isaac Newton wrote in his Common Place Book, "A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding." [1] Although the Dirac equation may seem more complicated than the Schrödinger equation, it has the advantage of being a physically realistic, and therefore comprehensible, description of nature. Besides its application to quantum mechanics, the Dirac formalism has been used by various researchers to describe classical wave dynamics. [2–12] Therefore we propose that the Dirac equation, which is both relativistic and describes spin angular momentum, is a better starting point for understanding quantum mechanics.

Although the Dirac equation is a deterministic description of the evolution of physical quantities, it is commonly used to calculate probabilities of various measurement outcomes. Bohmian mechanics, or pilot-wave theory, offers insight into the relationships between deterministic wave processes and quantum statistics. [13–15] Pilot-waves exhibiting quantum statistics have been experimentally demonstrated using silicone droplets bouncing on a vibrating water tank. [16–22] In this paper, however, we will only analyze the physical dynamics of wave evolution, and not the probabilistic nature of measurements.

We first expand on previous work in deriving a Dirac equation for spin density from the classical model of an ideal elastic solid. [8–10] We then present plane wave solutions, and use these as the basis for explanations of spatial reflection and special relativity. Next, we construct a Lagrangian and derive the dynamical operators of relativistic quantum mechanics. Finally, we show how wave interactions can yield the Pauli exclusion principle and interaction potentials.

2. METHODS: DERIVING AN EQUATION FOR SPIN DENSITY

2.1. Ideal Elastic Solid

We consider the case of an isotropic, homogeneous solid with a linear relationship between infinitesimal stress and strain. The usual expression for potential energy is (e.g. Ref. 23):

$$\int U \, d^3\mathbf{r} = \int \left(\frac{1}{2}\lambda(\nabla \cdot \boldsymbol{\xi})^2 + \mu e_{ij} e_{ij} \right) d^3\mathbf{r} \tag{1}$$

where $\boldsymbol{\xi}$ represents displacement, $e_{ij} = (\partial_i \xi_j + \partial_j \xi_i)/2$ is the symmetric strain tensor, and λ and μ are the Lame' parameters. This expression has the drawback that it does not cleanly separate compressible and rotational motion. We can remedy this as follows:

Expanding the square of the symmetrical strain tensor yields:

$$\begin{aligned} e_{ij}e_{ij} &= (\partial_x\xi_x)^2 + (\partial_y\xi_y)^2 + (\partial_z\xi_z)^2 \\ &+ \frac{1}{2}\left((\partial_x\xi_y + \partial_y\xi_x)^2 + (\partial_y\xi_z + \partial_z\xi_y)^2 + (\partial_z\xi_x + \partial_x\xi_z)^2 \right). \end{aligned} \tag{2}$$

Add $2(\partial_x\xi_x\partial_y\xi_y + \partial_y\xi_y\partial_z\xi_z + \partial_z\xi_z\partial_x\xi_x)$ to the first term and subtract it from the second term to obtain:

$$\begin{aligned} e_{ij}e_{ij} &= (\nabla \cdot \boldsymbol{\xi})^2 \\ &+ \frac{1}{2}\left((\partial_x\xi_y + \partial_y\xi_x)^2 + (\partial_y\xi_z + \partial_z\xi_y)^2 + (\partial_z\xi_x + \partial_x\xi_z)^2 \right) \\ &- 2(\partial_x\xi_x\partial_y\xi_y + \partial_y\xi_y\partial_z\xi_z + \partial_z\xi_z\partial_x\xi_x). \end{aligned} \tag{3}$$

Integrate the extra terms by parts on each of the two derivatives (assuming no contribution at infinity) to obtain:

$$\begin{aligned} e_{ij}e_{ij} &\to (\nabla \cdot \boldsymbol{\xi})^2 \\ &+ \frac{1}{2}\left((\partial_x\xi_y + \partial_y\xi_x)^2 + (\partial_y\xi_z + \partial_z\xi_y)^2 + (\partial_z\xi_x + \partial_x\xi_z)^2 \right) \\ &- 2(\partial_x\xi_y\partial_y\xi_x + \partial_y\xi_z\partial_z\xi_y + \partial_z\xi_x\partial_x\xi_z). \end{aligned} \tag{4}$$

This is equivalent to:

$$e_{ij}e_{ij} \to (\nabla\cdot\boldsymbol{\xi})^2 + \frac{1}{2}(\nabla\times\boldsymbol{\xi})^2\,. \tag{5}$$

The potential energy density may therefore be expressed as:

$$U = \frac{1}{2}(\lambda+2\mu)(\nabla\cdot\xi)^2 + \frac{1}{2}\mu(\nabla\times\boldsymbol{\xi})^2. \tag{6}$$

This form of the potential energy density separates infinitesimal irrotational and incompressible motion. It is a quadratic function of the first derivatives of displacement. The Lagrangian for infinitesimal incompressible motion is:

$$\mathcal{L} = \int \left(\frac{1}{2}\rho(\partial_t\boldsymbol{\xi})^2 - \frac{1}{2}\mu(\nabla\times\boldsymbol{\xi})^2\right) dV. \tag{7}$$

The Euler-Lagrange equation is the usual equation for infinitesimal shear waves:

$$\partial_t^2\boldsymbol{\xi} = -\frac{\mu}{\rho}\nabla\times\nabla\times\boldsymbol{\xi} \tag{8}$$

for which the wave speed is $c = \sqrt{\mu/\rho}$.

The incompressible potential energy in Eq. 7 was used by MacCullagh in 1837 to derive Eq. 8 as a description of light waves. [24]

2.2. Spin Angular Momentum

It is well known that elastic waves in solids have two types of momentum: that of the medium ($\rho\partial_t\boldsymbol{\xi}$) and that of the wave: $\rho(\nabla\xi_j)\partial_t\xi_j$ (see e.g. Ref. 26). Clearly there must also be two types of angular momentum in an elastic solid: "spin" associated with rotation of the medium, and "orbital" associated with rotation of the wave. However, spin angular momentum has not historically been considered to be a classical physics concept.

The key to understanding classical spin angular momentum is the Helmholtz decomposition of momentum density. The momentum density $\mathbf{p} = \rho\mathbf{v}$ consists of an incompressible (or rotational) part ($\tilde{\mathbf{p}}$), an irrotational (or compressible) part ($\check{\mathbf{p}}$), and a constant part ($\bar{\mathbf{p}}$) determined by the Helmholtz decomposition:

$$\mathbf{p} = \tilde{\mathbf{p}} + \check{\mathbf{p}} + \bar{\mathbf{p}} = \frac{1}{2}\nabla\times\mathbf{s} - \nabla\Phi + \bar{\mathbf{p}} \tag{9}$$

where:

$$\mathbf{s}(\mathbf{r},t) = \frac{1}{2\pi}\nabla\times\int_V \frac{\mathbf{p}(\mathbf{r}',t)-\bar{\mathbf{p}}}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{10a}$$

$$\Phi(\mathbf{r},t) = -\frac{1}{4\pi}\nabla\cdot\int_V \frac{\mathbf{p}(\mathbf{r}',t)-\bar{\mathbf{p}}}{|\mathbf{r}-\mathbf{r}'|}dV'\,. \tag{10b}$$

Previous work has demonstrated that $\mathbf{s}$ represents angular momentum density corresponding to spin in relativistic quantum mechanics. [8–10] Hence we refer to $\mathbf{s}$ as "spin density".

Assuming sufficiently rapid fall-off at large distances, the volume integral of spin density is equal to the volume integral of the first moment of momentum $\mathbf{r}\times\tilde{\mathbf{p}}$. The two representations of angular momentum density are related by integration by parts [10]:

$$\begin{aligned}
&\int \mathbf{r}\times\frac{1}{2}(\nabla\times\mathbf{s})d^3r = \frac{1}{2}\int\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \mathbf{r}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{r}\right)d^3r \\
&= \frac{1}{2}\int\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \partial_i(r_i\mathbf{s}) + \mathbf{s}(\nabla\cdot\mathbf{r}) - \mathbf{s}\cdot\nabla\mathbf{r}\right)d^3r \\
&= \int \mathbf{s}\,d^3r\,.
\end{aligned} \tag{11}$$

where the total derivatives are assumed not to contribute to the last line, since they can be converted into surface integrals that are assumed to vanish.

Unlike the "moment of momentum" definition, spin angular momentum density is an intrinsic property defined at each point in space. Coordinate-independent descriptions of rotational dynamics can actually be traced back to the nineteenth century.[27] In 1891 Oliver Heaviside recognized MacCullagh's force density in Eq. 8 as being the curl of a torque density that is proportional to an infinitesimal rotation angle $\boldsymbol{\Theta} = (1/2)\nabla \times \boldsymbol{\xi}$. [28] However, this idea seems to have been largely forgotten. Students should be encouraged to ponder how physics might have developed differently had a simple interpretation of spin angular momentum been available to the early pioneers of quantum mechanics.

The rotational kinetic energy is: [10]

$$\begin{aligned} K &= \frac{1}{2\rho}\int \tilde{p}^2 d^3r = \frac{1}{2\rho}\int \left[\frac{1}{2}\nabla \times \mathbf{s}\right]^2 dV \\ &= \frac{1}{8\rho}\int \left[\mathbf{s}\cdot[\nabla\times(\nabla\times\mathbf{s})] + \nabla\cdot(\mathbf{s}\times(\nabla\times\mathbf{s}))\right] dV \\ &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, dV, \end{aligned} \tag{12}$$

where $\mathbf{w} = \nabla \times \mathbf{v}/2$ is the angular velocity (sometimes confusingly referred to as "spin" in the literature). In this case the divergence term is assumed not to contribute to the volume integral, since it can be converted into a surface integral.

According to Eq. 12, spin density ($\mathbf{s}$) is the momentum conjugate to angular velocity:

$$\frac{\delta}{\delta w_i}\int \frac{1}{2} w_j s_j\, dV = \frac{1}{2}\int \left(\frac{\delta w_j}{\delta w_i} s_j + w_j \frac{\delta s_j}{\delta w_i}\right) dV = \frac{1}{2}s_i + \frac{1}{2}s_i = s_i\,, \tag{13}$$

where integration by parts was used twice to evaluate the second term in the integral.

As an example, consider a cylinder of radius R aligned with the z-axis and rotating rigidly with angular velocity w_0. The motion is described by these non-zero variables: [10]

$$s_z = \rho w_0[R^2 - r^2] \text{ for } r \le R \text{ and zero for } r > R; \tag{14a}$$

$$v_\phi = -\frac{1}{2\rho}\frac{\partial}{\partial r}s_z = r w_0 \text{ for } r \le R; \text{ and zero for } r > R; \tag{14b}$$

$$w_z = \frac{1}{2r}\frac{\partial}{\partial r} r v_\phi = w_0\left[1 - R\delta(r-R)/2\right] \text{ for } r \le R; \text{ and zero for } r > R. \tag{14c}$$

The total angular momentum per unit height is

$$\begin{aligned} S_z &= 2\pi\int_0^R s_z r dr = 2\pi\int_0^R \rho w_0[R^2 - r^2] r dr \\ &= \frac{1}{2}\pi\rho R^4 w_0 = \frac{1}{2}MR^2 w_0 \\ &= I w_0\,, \end{aligned} \tag{15}$$

where we have used the mass per unit height $M = \rho\pi R^2$ and moment of inertia per unit height $I = MR^2/2$.

The kinetic energy per unit height is

$$\begin{aligned} K &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, r dr d\phi = \pi\int_0^R w_0\left[1 - R\delta(r-R)/2\right]\rho w_0[R^2 - r^2] r dr \\ &= \pi\rho w_0^2\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{MR^2}{4}w_0^2 = \frac{1}{2}I w_0^2\,. \end{aligned} \tag{16}$$

These are in agreement with standard rotational dynamics. Students should understand that spin angular momentum is well-defined in classical physics.

Defining $\boldsymbol{\Theta} = (1/2)\nabla \times \boldsymbol{\xi}$, Eq. 8 becomes:

$$\partial_t(\nabla \times \mathbf{s}) + 4\mu\nabla \times \boldsymbol{\Theta} = 0. \tag{17}$$

Note that $\boldsymbol{\Theta}$ is a vector and only represents an angle for infinitesimal motion. Assuming $\nabla \cdot \mathbf{s} = 0$, the Helmholtz decomposition yields:

$$\partial_t \mathbf{s} = -4\mu\boldsymbol{\Theta}. \tag{18}$$

This equation states that the rate of change of angular momentum density is equal to torque density, which is proportional to infinitesimal rotation angle.

The next step is to relate the displacement $\boldsymbol{\xi}$ to the spin density $\mathbf{s}$. For infinitesimal motion, define a vector potential $\mathbf{Q}$ such that $\partial_t \mathbf{Q} = \mathbf{s}$. Since the curl of $\mathbf{s}$ is proportional to velocity, the curl of $\mathbf{Q}$ must be proportional to displacement:

$$\frac{1}{2\rho}\nabla \times \mathbf{Q} = \boldsymbol{\xi}\,. \tag{19}$$

Therefore the equation for $\mathbf{s}$ is equivalent to:

$$\partial_t^2 \mathbf{Q} + c^2 \nabla \times \nabla \times \mathbf{Q} = 0\,, \tag{20}$$

where $c^2 = \mu/\rho$. The curl of this equation yields Eq. 8.

Thus far we have assumed infinitesimal motion. Previous work attempted to describe finite motion by adding convection and rotation terms: [8–10]

$$\partial_t \mathbf{s} + \mathbf{v} \cdot \nabla \mathbf{s} - \mathbf{w} \times \mathbf{s} = -c^2 \nabla \times \nabla \times \mathbf{Q} = \boldsymbol{\tau}. \tag{21}$$

The logic of this equation is that changes attributable to translation ($\mathbf{v} \cdot \nabla \mathbf{s}$) and rotation ($-\mathbf{w} \times \mathbf{s}$) do not require torque density ($\boldsymbol{\tau}$). Similarly, the momentum density equation may be interpreted as a statement that changes due to translation ($\mathbf{v} \cdot \nabla \mathbf{p}$) do not require force density ($\mathbf{f}$):

$$\partial_t \mathbf{p} + \mathbf{v} \cdot \nabla \mathbf{p} = \mathbf{f}. \tag{22}$$

However, the validity of Eq. 21 is unclear. A more rigorous approach is to apply the Helmholtz decomposition to the momentum density convection term $\mathbf{v} \cdot \nabla \mathbf{p} = (1/2)\rho \nabla v^2 + 2\mathbf{w} \times \mathbf{p}$. Assuming no contributions from boundary integrals, this yields the equation:

$$\partial_t \mathbf{s} + \nabla \times \frac{1}{\pi} \int \frac{\mathbf{w}' \times \mathbf{p}'}{|\mathbf{r} - \mathbf{r}'|} dV' = \boldsymbol{\tau}. \tag{23}$$

In terms of the vector potential $\mathbf{Q}$:

$$\partial_t^2 \mathbf{Q} + c^2 \nabla \times \nabla \times \mathbf{Q} = -\nabla \times \frac{1}{\pi} \int \frac{\mathbf{w}' \times \mathbf{p}'}{|\mathbf{r} - \mathbf{r}'|} dV'\,. \tag{24}$$

In this paper we only consider infinitesimal motion so that the nonlinear term is neglected.

2.3. Dirac Equation

Eq. 20 is a second-order vector equation. In order to compute the evolution of physical quantities, an equation of first-order in time derivatives is required. We will follow Refs. [8] and [10] by starting with one-dimensional waves and then generalizing to three dimensions.

2.3.1. One-Dimensional Waves

Consider a one-component wave propagating in one-dimension with amplitude of $a(z,t)$. If the wave equation is

$$\partial_t^2 a = c^2 \partial_z^2 a\,, \tag{25}$$

then the general solution consists of backward (B) and forward (F) propagating waves:

$$a = a_B(ct + z) + a_F(ct - z)\,. \tag{26}$$

The two directions of wave propagation are clearly independent states, and they are separated in space by a 180° rotation. This property is the fundamental characteristic of spin one-half states. Generalization to three dimensional space should therefore involve spinor wave functions.

The forward and backward waves satisfy the equations:

$$\begin{aligned} \partial_t a_B &= \partial_z a_B \\ \partial_t a_F &= -\partial_z a_F\,. \end{aligned} \tag{27}$$

Defining $\dot{a} = \partial_t a$, we can write the wave equation as a first-order matrix equation:

$$\partial_t \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} c\partial_z \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} = 0\,. \tag{28}$$

We have thus achieved our goal of converting a one-dimensional second-order wave equation into a first-order matrix equation. Generalization to three dimensional vector waves requires additional components. One possibility is to introduce vector components such as (a_{Bi}) and (a_{Fi}) to make a 6-element column vector in the equation above. Unfortunately, this method does not allow a simple means for changing the direction of the derivative (∂_z). Therefore we follow a different path.

First, note that the procedure above specifies independent components with positive and negative wave velocity, and uses a diagonal matrix to relate spatial and temporal derivatives. We can apply a similar technique to separate positive and negative values of the wave function. Letting a_B and a_F represent the z-components of vectors, separate each component of the wave into positive and negative parts ($\dot{a}_B = \dot{a}_{B+} - \dot{a}_{B-}$ and $\dot{a}_F = \dot{a}_{F+} - \dot{a}_{F-}$) so that each of the four wave components $(\dot{a}_{B+}, \dot{a}_{B-}, \dot{a}_{F+}, \dot{a}_{F-})$ is positive-definite. With these definitions, we have:

$$\dot{a} = \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = \psi^T \sigma_3 \psi \tag{29}$$

where σ_3 represents the 4×4 Dirac matrix for the z-component of spin density, and the the four-component column vectors are called Dirac bispinors. In one dimension, the significance of simultaneous positive and negative components is unclear. We will see that in three dimensions, simultaneous positive and negative components for one direction indicates polarization in a different direction.

The spatial derivative is now given by:

$$c\partial_z a = -\begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = -\psi^T \gamma^5 \psi \tag{30}$$

where an overall minus sign has been introduced in order to maintain consistency with the chiral representation of Dirac wave functions. The matrix γ^5 is the Dirac matrix for chirality. If the amplitude (a) represents rotation angle, then positive and negative chirality $(\partial_z a)$ are analogous to right- and left-handed threads on a screw. The chirality projection operators are:

$$\begin{aligned} \frac{1}{2}(I - \gamma^5)\psi &\equiv \psi_L \\ \frac{1}{2}(I + \gamma^5)\psi &\equiv \psi_R \end{aligned} \tag{31}$$

Wave velocity (v) is obtained by combining the two matrices used above:

$$v\psi = c \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = c\gamma^5 \sigma_3 \psi\,. \tag{32}$$

The one-dimensional wave equation may be written in the form:

$$\partial_t[\psi^T \sigma_3 \psi] + c\partial_z[\psi^T \gamma^5 \psi] = \partial_t^2 a - c^2 \partial_z^2 a = 0\,. \tag{33}$$

Other matrices may be inserted between the wave functions, resulting in the following corresponding expressions (correcting a mistake in Ref. 10). Each of these is equal to zero for the wave solutions:

$$\partial_t[\psi^T\psi] + c\partial_z[\psi^T\gamma^5\sigma_3\psi] = \partial_t|\partial_t a_F| + \partial_t|\partial_t a_B| + c^2\left(\partial_z|\partial_z a_F| - \partial_z|\partial_z a_B|\right); \tag{34a}$$
$$\partial_t[\psi^T\gamma^5\sigma_3\psi] + c\partial_z[\psi^T\psi] = c\left(\partial_t|\partial_z a_F| - \partial_t|\partial_z a_B| + \partial_z|\partial_t a_F| + \partial_z|\partial_t a_B|\right); \tag{34b}$$
$$\partial_t[\psi^T\gamma^5\psi] + c\partial_z[\psi^T\sigma_3\psi] = \partial_t[-c\partial_z a] + c\partial_z[\partial_t a]\,. \tag{34c}$$

The one-dimensional Dirac equation is itself useful for teaching purposes. [29, 30] However, its equivalence with the one-dimensional second-order wave equation has not been widely recognized. Next we will show how to generalize the first-order equation to three spatial dimensions.

2.3.2. Three-Dimensional Vector Waves

Generalization to three dimensions is based on the fact that the matrix σ_3 may be regarded as representing one component of a three-dimensional vector. An arbitrary vector $\mathbf{a} = (a_x, a_y, a_z)$ may be written in terms of a 2-component complex spinor η and the Pauli spin matrices $\boldsymbol{\sigma}^P = (\sigma_1^P, \sigma_2^P, \sigma_3^P)$ as:

$$\begin{aligned} a_x &= \eta^\dagger\sigma_1^P\eta = \eta^\dagger\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eta\,, \\ a_y &= \eta^\dagger\sigma_2^P\eta = \eta^\dagger\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\eta\,, \\ a_z &= \eta^\dagger\sigma_3^P\eta = \eta^\dagger\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\eta\,. \end{aligned} \tag{35}$$

The normalized spinor eigenfunctions for each direction are:

$$\begin{aligned} \sigma_1^P\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \quad & \sigma_1^P\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} &= -\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}; \\ \sigma_2^P\begin{bmatrix} 1/\sqrt{2} \\ \mathrm{i}/\sqrt{2} \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} \\ \mathrm{i}/\sqrt{2} \end{bmatrix}; \quad & \sigma_2^P\begin{bmatrix} 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} \end{bmatrix} &= -\begin{bmatrix} 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} \end{bmatrix}; \\ \sigma_3^P\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad & \sigma_3^P\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= -\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{36}$$

The algebra of the Pauli matrices is called "geometric algebra":

$$\begin{aligned} \sigma_1^P\sigma_2^P\sigma_3^P &= \mathrm{i}I\,, \\ \sigma_i^P\sigma_j^P &= \delta_{ij}I + \mathrm{i}\epsilon_{ijk}\sigma_k^P\,. \end{aligned} \tag{37}$$

where the unit imaginary "i" represents a unit oriented "volume". The $\boldsymbol{\sigma}^P$ matrices may in general represent axial or polar vectors, but they are most commonly associated with spin density, which is an axial vector. The fourth independent matrix in this algebra is the identity matrix (I). The direction of the vector $\eta^\dagger\boldsymbol{\sigma}^P\eta$ can be rotated by an arbitrary angle ϕ about an axis $\hat{\mathbf{e}}_\phi$ using operations of the form (with $\boldsymbol{\phi} = \phi\hat{\mathbf{e}}_\phi$):

$$R_{\boldsymbol{\phi}}(\eta^\dagger\boldsymbol{\sigma}^P\eta) = \eta^\dagger R^{-1}(\boldsymbol{\phi})\boldsymbol{\sigma}^P R(\boldsymbol{\phi})\eta = \eta^\dagger\exp\left(\mathrm{i}\boldsymbol{\sigma}^P\cdot\boldsymbol{\phi}/2\right)\boldsymbol{\sigma}^P\exp\left(-\mathrm{i}\boldsymbol{\sigma}^P\cdot\boldsymbol{\phi}/2\right)\eta. \tag{38}$$

The Dirac wave functions specify not a single vector, but spatial and temporal derivatives of a vector. Forward and backward waves along each axis are combined by replacing the Pauli matrices with the corresponding Dirac spin matrices and replacing the two-component spinor η with a 4-component bispinor ψ. In terms of the Pauli matrices, the 4×4 Dirac spin matrices are:

$$\sigma_1 = \begin{pmatrix} \sigma_1^P & \mathbf{0} \\ \mathbf{0} & \sigma_1^P \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \sigma_2^P & \mathbf{0} \\ \mathbf{0} & \sigma_2^P \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \sigma_3^P & \mathbf{0} \\ \mathbf{0} & \sigma_3^P \end{pmatrix} \tag{39}$$

where $\mathbf{0}$ is the 2×2 null matrix.

Just as there are three Pauli matrices indicating different vector directions, there are also three orthogonal matrices associated with wave velocity. In the chiral notation, these are:

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \gamma^6 = \begin{pmatrix} \mathbf{0} & -\mathrm{i}\mathbf{I} \\ \mathrm{i}\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \gamma^5 = -\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \tag{40}$$

where $\mathbf{I}$ is the 2×2 identity matrix. The γ-matrices above have the same form as the Pauli spin matrices except that the set $(\gamma^0, \gamma^5, \gamma^6)$ corresponds to $(\sigma_1, -\sigma_3, \sigma_2)$. The matrix γ^6 (identified as $-\gamma^4$ in prior publications [8, 10]) is defined so that $\gamma^0\gamma^5 = \mathrm{i}\gamma^6$.

The one-dimensional wave equation (Eq. 33) has the bispinor form:

$$\psi^T \left\{ \sigma_3 \partial_t \psi + c\gamma^5 \partial_z \psi \right\} + \text{Transpose} = 0\,. \tag{41}$$

We can separate a common factor of $\psi^\dagger \sigma_3$:

$$\psi^\dagger \sigma_3 \left\{ \partial_t \psi + c\gamma^5 \sigma_3 \partial_z \psi \right\} + \text{Transpose} = 0\,. \tag{42}$$

For arbitrary vector components and derivatives, the matrices and derivatives are generalized to arbitrary directions, and the bispinor wave functions are allowed to be complex:

$$\psi^\dagger \sigma_i \left\{ \partial_t \psi + c\gamma^5 \sigma_j \partial_j \psi \right\} + \text{adjoint} = 0\,. \tag{43}$$

This is the first-order wave equation for vector waves in three dimensions.

Expanding the spatial derivative term in Eq. 43 yields the 3-D generalization of the wave equation (Eq. 33):

$$0 = \partial_t \left[\psi^\dagger \boldsymbol{\sigma} \psi \right] + c\nabla \left[\psi^\dagger \gamma^5 \psi \right] - \mathrm{i}c \left\{ \left[\nabla \psi^\dagger \right] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi \right\}\,. \tag{44}$$

This corresponds, with corresponding terms in order, to the vector wave equation:

$$0 = \partial_t^2 \mathbf{a} - c^2 \nabla(\nabla \cdot \mathbf{a}) + c^2 \nabla \times (\nabla \times \mathbf{a})\,. \tag{45}$$

This is the result we have been seeking. We have rewritten the second-order vector wave equation as a first order equation involving Dirac bispinors. The validity of this correspondence, which we will confirm with examples, demonstrates that the Dirac equation of relativistic quantum mechanics is simply a special case of an ordinary vector wave equation.

Replacing the vector $\mathbf{a}$ by $2\mathbf{Q}$ yields the following physical correspondences:

$$\mathbf{s} = \partial_t \mathbf{Q} \equiv \frac{1}{2} \left[\psi^\dagger \boldsymbol{\sigma} \psi \right]; \tag{46a}$$

$$c\nabla \cdot \mathbf{Q} \equiv -\frac{1}{2} \left[\psi^\dagger \gamma^5 \psi \right]; \tag{46b}$$

$$c^2 \left\{ \nabla \times \nabla \times \mathbf{Q} \right\} \equiv -\frac{\mathrm{i}c}{2} \left\{ \left[\nabla \psi^\dagger \right] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi \right\}; \tag{46c}$$

$$0 = \frac{\mathrm{i}c}{2} \nabla \cdot \left\{ \left[\nabla \psi^\dagger \right] \times \gamma^5 \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi \right\}\,. \tag{46d}$$

These identifications provide seven independent constraints on the eight free parameters of the complex Dirac bispinor: three for the first, one for the second, two for the third (since a curl has only two independent components), and one for the fourth. There is also an arbitrary overall phase factor. The last identification simply states that the divergence of a curl is zero. This condition is necessary for consistency.

The first-order wave equation (Eq. 43) can be reduced to:

$$\partial_t \psi + c\gamma^5 \boldsymbol{\sigma} \cdot \nabla \psi + \mathrm{i}\chi \psi = 0\,, \tag{47}$$

where χ is any operator with the property

$$\mathrm{Re} \left\{ \psi^\dagger \sigma_j \mathrm{i}\chi \psi \right\} = 0\,. \tag{48}$$

The equation for a free electron is obtained by the choosing $\chi = \Omega \gamma^0$ with $\Omega = m_e c^2/\hbar$. Hence the Dirac equation for an electron may be interpreted as an ordinary wave equation with a clear dynamical interpretation describing the motion of an elastic solid.

According to the above analysis, the first-order Dirac equation is a kind of factorization (or square root) of a second-order wave equation. Others have made different factorizations using multivariate 4-vectors, quaternions, or octonions. [31–34]

Multiplying Eq. 47 by $\psi^\dagger$ and adding the adjoint yields a conservation law with density $\psi^\dagger\psi$ and current $\psi^\dagger c\gamma^5\boldsymbol{\sigma}\psi$:

$$\partial_t(\psi^\dagger\psi) + \nabla\cdot(\psi^\dagger c\gamma^5\boldsymbol{\sigma}\psi) = 0\,. \tag{49}$$

In quantum mechanics this equation is regarded as a conservation law for probability density, but in both classical and quantum mechanics it is part of the description of the evolution of spin angular momentum density. It is the three-dimensional generalization of Eq. 34a.

3. RESULTS: APPLICATIONS OF CLASSICAL SPIN DENSITY

3.1. Sample Plane Wave Solutions

As a simple mathematical example, the longitudinal wave $(Q_x, Q_y, Q_z) = (0, 0, Q_0\sin(\omega t - kz))$ propagating along the z-axis may be expressed in the bispinor form:

$$\psi = \sqrt{2\omega Q_0}\exp[-\mathrm{i}(\omega t - kz)/2]\begin{bmatrix} 0 \\ \sin([\omega t - kz]/2) \\ \cos([\omega t - kz]/2) \\ 0 \end{bmatrix}. \tag{50}$$

The phase factor in front is introduced for later consistency with transverse waves. For $\omega = ck$, this wave function yields:

$$\mathbf{s} = \partial_t\mathbf{Q} = \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = (0, 0, \omega Q_0\cos(\omega t - kz))\,; \tag{51a}$$

$$c\nabla\cdot\mathbf{Q} = -\frac{1}{2}\left[\psi^\dagger\gamma^5\psi\right] = -\omega Q_0\cos(\omega t - kz)\,; \tag{51b}$$

$$c^2(\{\nabla\times\nabla\times\mathbf{Q}\}) = -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\} = (0, 0, 0)\,. \tag{51c}$$

In this case $\nabla\times\mathbf{s} = 0$, so this wave solution is not relevant for describing shear waves in an elastic solid. However, longitudinal waves are the classical analogues of quantum mechanical waves.

In addition to the wave variables described above, there are other "observables" that may be computed from the wave function. These include the vector quantites:

$$\psi^\dagger\gamma^0(\sigma_1, \sigma_2, \sigma_3)\psi = (\omega Q_0\sin(\omega t - kz), 0, 0)\,; \tag{52a}$$

$$\psi^\dagger\gamma^5(\sigma_1, \sigma_2, \sigma_3)\psi = (0, 0, \omega Q_0)\,; \tag{52b}$$

$$\psi^\dagger\gamma^6(\sigma_1, \sigma_2, \sigma_3)\psi = (0, \omega Q_0\sin(\omega t - kz), 0)\,. \tag{52c}$$

In this case the current $\psi^\dagger\gamma^5\sigma_3\psi$ is proportional to constant wave velocity, so $\gamma^5\sigma_3$ is associated with $\hat{\mathbf{z}}$. The other two "vectors" are evidently in orthogonal directions, but it is not immediately clear which is $\hat{\mathbf{x}}$ and which is $\hat{\mathbf{y}}$. The matrices $(\gamma^0, \gamma^5, \gamma^6)$ may be interpreted as defining directions relative to the wave velocity direction. Specifically, multiplication of the wave function by $\exp[-\mathrm{i}\gamma^0\phi_0/2]$ rotates the velocity about the y-axis by angle ϕ_0, provided that the variables (x, y, z) are also rotated accordingly ($kz \to k(z\cos\phi_0 + x\sin\phi_0)$). Likewise, multiplication of the original wave function by $\exp[\mathrm{i}\gamma^6\phi_6/2]$ rotates the wave velocity about the x-axis.

To obtain a linearly polarized transverse wave solution, we rotate the wave velocity independently of the polarization direction using the γ matrices.

For example, velocity rotation by $-\pi/2$ about the x-axis is performed by multiplying the wave function in Eq. 50 by $\exp[-\mathrm{i}\gamma^6(\pi/4)]$ and changing the wave direction $z \to y$, so that the bispinor becomes:

$$\psi = \sqrt{\omega Q_0}\exp[-\mathrm{i}(\omega t - ky)/2]\begin{bmatrix} -\cos([\omega t - ky]/2) \\ \sin([\omega t - ky]/2) \\ \cos([\omega t - ky]/2) \\ \sin([\omega t - ky]/2) \end{bmatrix}, \tag{53}$$

and the new wave vector potential is $(Q_x, Q_y, Q_z) = (0, 0, Q_0 \sin(\omega t - ky))$. The phase factor in front is now necessary for satisfaction of the Dirac equation, even though it does not affect the real-valued spin density vector field. The correspondence between matrices and coordinates is also unique. The wave velocity is now proportional to $-\psi^\dagger\gamma^0\sigma_3\hat{\mathbf{y}}\psi$. Changing the sign of the matrix γ^6, or changing y to any other wave direction, would produce a wave function that no longer satisfies the Dirac equation. Hence the γ-matrices have a definite handedness. This is an important fact to consider when analyzing spatial reflection.

Alternatively, velocity rotation of the bispinor in Eq. 50 by $\pi/2$ about the y-axis is performed by multiplying the wave function by $\exp[-\mathrm{i}\gamma^0(\pi/4)]$ and changing the wave direction $z \to x$, so that the bispinor becomes:

$$\psi = \sqrt{\omega Q_0}\exp\left[-\mathrm{i}\left(\omega t - kx\right)/2\right]\begin{bmatrix} -\mathrm{i}\,\cos\left([\omega t - kx]/2\right) \\ \sin\left([\omega t - kx]/2\right) \\ \cos\left([\omega t - kx]/2\right) \\ -\mathrm{i}\,\sin\left([\omega t - kx]/2\right) \end{bmatrix}, \tag{54}$$

and the new wave vector potential is $(Q_x, Q_y, Q_z) = (0, 0, Q_0 \sin(\omega t - kx))$.

Other wave variables are:

$$\mathbf{s} = \partial_t\mathbf{Q} = \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = (0, 0, \omega Q_0\cos(\omega t - kx))\,; \tag{55a}$$

$$c\nabla\cdot\mathbf{Q} = -\frac{1}{2}\left[\psi^\dagger\gamma^5\psi\right] = 0\,; \tag{55b}$$

$$\begin{aligned} c^2(\{\nabla\times\nabla\times\mathbf{Q}\}) &= -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\} \\ &= (0, 0, c^2k^2Q_0\sin(\omega t - kx))\,. \end{aligned} \tag{55c}$$

Arbitrary monochromatic plane waves can be obtained by suitable scaling, overall rotation, and velocity rotation operations. Two constants of the motion are:

$$\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi) = -\mathrm{Re}(\psi^\dagger\mathrm{i}c\gamma^5\boldsymbol{\sigma}\cdot\nabla\psi) = \omega^2Q_0\,. \tag{56}$$

The $b-$vectors are:

$$\psi^\dagger\gamma^0(\sigma_1, \sigma_2, \sigma_3)\psi = (\omega Q_0\sin(\omega t - kx), 0, 0)\,; \tag{57a}$$

$$\psi^\dagger\gamma^5(\sigma_1, \sigma_2, \sigma_3)\psi = (0, -\omega Q_0\sin(\omega t - kx), 0)\,; \tag{57b}$$

$$\psi^\dagger\gamma^6(\sigma_1, \sigma_2, \sigma_3)\psi = (0, 0, \omega Q_0)\,. \tag{57c}$$

Now the wave velocity is proportional to $\psi^\dagger\gamma^6\sigma_3\hat{\mathbf{x}}\psi$.

Table I shows how wave velocity and spin directions are related. Starting with a longitudinal wave propagating in the z-direction, wave velocity is rotated using the γ^0 and γ^6 matrices as described above. The resulting wave velocity matrices and unit vectors are listed in the "Initial" column.

Note that the triplet $(-\gamma^0\sigma_3\hat{\mathbf{y}}, \gamma^5\sigma_3\hat{\mathbf{z}}, \gamma^6\sigma_3\hat{\mathbf{x}})$ forms a left-handed coordinate system. So while $\exp(-\mathrm{i}\sigma_1\varphi/2)$ represents rotation by φ about the x-axis, the operator $\exp(-\mathrm{i}\gamma^0\varphi/2)$ represents a rotation of wave velocity by $-\varphi$ about the axis associated with γ^0 (and similarly for γ^5 and γ^6). For example, the y-component of wave velocity is initially represented by $-\gamma^0\sigma_3\hat{\mathbf{y}}$, but rotation of wave velocity about the y-axis was performed using the operator $\exp(-\mathrm{i}\gamma^0\pi/4)$ rather than $\exp(+\mathrm{i}\gamma^0\pi/4)$.

TABLE I. Spin and wave velocity operators resulting from 90° rotations.

Rotation Axis:	Initial	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{y}}$ then $\hat{\mathbf{z}}$	$\hat{\mathbf{z}}$ then $\hat{\mathbf{y}}$
Rotation Operator:	None	$e^{-\mathrm{i}\sigma_1\frac{\pi}{4}}$	$e^{-\mathrm{i}\sigma_2\frac{\pi}{4}}$	$e^{-\mathrm{i}\sigma_3\frac{\pi}{4}}$	$e^{-\mathrm{i}\sigma_3\frac{\pi}{4}}e^{-\mathrm{i}\sigma_2\frac{\pi}{4}}$	$e^{-\mathrm{i}\sigma_2\frac{\pi}{4}}e^{-\mathrm{i}\sigma_3\frac{\pi}{4}}$
Change of variable:	None	$z \to -y$	$z \to x$	$z \to z$	$z \to y$	$z \to x$
Final Spin Axis:	$\sigma_3\hat{\mathbf{z}}$	$-\sigma_2\hat{\mathbf{y}}$	$\sigma_1\hat{\mathbf{x}}$	$\sigma_3\hat{\mathbf{z}}$	$\sigma_2\hat{\mathbf{y}}$	$\sigma_1\hat{\mathbf{x}}$
Wave	$\gamma^6\sigma_3\hat{\mathbf{x}}$	$\gamma^6\sigma_2\hat{\mathbf{x}}$	$\gamma^5\sigma_1\hat{\mathbf{x}}$	$\gamma^0\sigma_3\hat{\mathbf{x}}$	$\gamma^0\sigma_2\hat{\mathbf{x}}$	$\gamma^5\sigma_1\hat{\mathbf{x}}$
Velocity	$-\gamma^0\sigma_3\hat{\mathbf{y}}$	$-\gamma^5\sigma_2\hat{\mathbf{y}}$	$-\gamma^0\sigma_1\hat{\mathbf{y}}$	$\gamma^6\sigma_3\hat{\mathbf{y}}$	$\gamma^5\sigma_2\hat{\mathbf{y}}$	$\gamma^6\sigma_1\hat{\mathbf{y}}$
Operators	$\gamma^5\sigma_3\hat{\mathbf{z}}$	$-\gamma^0\sigma_2\hat{\mathbf{z}}$	$-\gamma^6\sigma_1\hat{\mathbf{z}}$	$\gamma^5\sigma_3\hat{\mathbf{z}}$	$-\gamma^6\sigma_2\hat{\mathbf{z}}$	$-\gamma^0\sigma_1\hat{\mathbf{z}}$

Note that the longitudinal wave propagation direction is always represented by $\gamma^5\sigma_s\hat{\mathbf{s}}$, where s represents the spin direction. This is consistent with the interpretation of $-\psi^\dagger\gamma^5\psi/2$ as the divergence of the vector potential $\mathbf{Q}$ (the derivative along the direction of the vector). The matrix σ_s associated with the spin direction is also associated with all of the wave velocity directions for the given spin polarization. The spin rotation operators use the spin matrices associated with the original fixed axes, with the operations performed in order from right to left. For a given spin direction, the wave velocity is rotated using the γ matrices with corresponding changes of coordinate variables.

Waves may also be rotated using Lorentz boosts. Starting with the longitudinal wave in the z-direction, a Lorentz boost in the x-direction yields:

$$\psi'(z',t) = \exp\left(\gamma^5\sigma_1\alpha/2\right)\psi(z',t) = \cosh\left(\alpha/2\right)\psi(z',t) + \sinh\left(\alpha/2\right)\gamma^5\sigma_1\,\psi(z',t) \tag{58}$$

For the boost parameter $\alpha\to\infty$ and corresponding coordinate change $z'=x$, this can be normalized to yield:

$$\psi' = (\psi+\gamma^5\sigma_1\,\psi)/\sqrt{2} = \sqrt{\omega Q_0}\exp\left[-\mathrm{i}\left(\omega t-kx\right)/2\right]\begin{bmatrix} -\sin\left([\omega t-kx]/2\right) \\ \sin\left([\omega t-kx]/2\right) \\ \cos\left([\omega t-kx]/2\right) \\ \cos\left([\omega t-kx]/2\right)\end{bmatrix}, \tag{59}$$

for which the wave vector potential is the longitudinal wave $(Q_x,Q_y,Q_z) = (Q_0\sin\left(\omega t-kx\right),0,0)$.

3.2. Spatial Reflection

We have established that the γ-matrices together form a vector space of orthogonal directions. These matrices have exactly the same algebra as the σ-matrices, and their role in wave velocity rotations is completely analogous to the role of σ-matrices in general rotations. For plane waves, each γ-matrix is associated with a unique direction in space. However, unlike spin density which is an axial vector with positive parity, spatial derivatives and wave velocity are polar vectors with negative parity. Hence spatial reflection must change the sign of physical quantities computed from the γ-matrices. The quantum mechanical "parity" operator does not do that, since it does not invert quantities calculated from γ^0:

$$P\psi(\mathbf{r},t) = \gamma^0\psi(-\mathbf{r},t)\,. \tag{60}$$

This is a fundamental difference between quantum mechanics and classical wave theory. The reason is that the quantum mechanical parity operator is based on an assumption that the underlying physical spacetime is a Minkowski space, whereas classical physics assumes that physical spacetime is Galilean, and the Minkowski space of measurements is the result of making measurements using waves that propagate at the speed of light. [35, 44]

Previous work argued that spatial reflection must be equivalent to inverting all three of the γ-matrices without inverting the σ-matrices, but lacked a detailed prescription for accomplishing that by operations on the wave function. [8, 35]. The main difficulty is that complex conjugation (ψ^*) inverts both γ^6 and σ_2. Hence the operator $\gamma^6\psi^*$ not only inverts all of the γ matrices, it also inverts σ_2. It does not seem possible to distinguish between $iI=\sigma_1\sigma_2\sigma_3$ and $iI=\gamma_0\gamma_5\gamma_6$.

One possible solution to this dilemma is based on the fact that longitudinal spin waves have no physical significance. This includes quantities calculated from γ^5 and $\gamma^5\sigma_s$. The curl involves the matrices $\gamma^5\sigma_{\bar{s}}$, where $\bar{s}$ indicates any direction perpendicular to the spin direction. These should be inverted. Therefore a possible classical spatial reflection operator (P') could be:

$$P'\psi(\mathbf{r},t) = \gamma^5\sigma_s\psi(-\mathbf{r},t)\,. \tag{61}$$

This operator leaves spin unchanged but inverts all of the transverse wave velocities and the curl of the vector potential $\mathbf{Q}$. In addition to the aforementioned matrices, it also does not invert quantities computed from $\gamma^{\bar{5}}\sigma_{\bar{s}}$, where the bar over the "5" indicates any other index. For our plane waves with $\mathbf{Q}(\mathbf{r},t)\sim\sin\left(\omega t-kz\right)$, these quantities have the same functional dependence as $\mathbf{Q}$ and therefore do not represent spatial derivatives of the vector potential. Hence it seems reasonable (though not proven here) that these quantities should not be inverted by spatial reflection.

Compare this operator for spatial reflection with the standard PCT operator, which within an arbitrary phase factor is:

$$PCT\psi(\mathbf{r},t) = \gamma^5\psi(-\mathbf{r},-t)\,. \tag{62}$$

For spin eigenfunctions ($\sigma_s\psi \propto \psi$), the only difference between the classical parity operator and the standard PCT operator is the sign of the time in the functional argument. The Feynman-Stueckelberg interpretation of antimatter describes positrons as "negative-energy electrons running backward in space-time", with the positive-energy positron wave function $\psi_+(E;\mathbf{r},t)$ related to a negative-energy electron wave function $\psi_-(-E;\mathbf{r},t)$ by: [36]

$$\psi_+(E;\mathbf{r},t) = PCT\psi_-(-E;\mathbf{r},t) = \gamma^5\psi_-(-E;-\mathbf{r},-t)\,. \tag{63}$$

For any oscillation or wave with a factor of ωt in the phase, reversing the sign of the time is equivalent to reversing the sign of the frequency, which is proportional to energy in quantum mechanics. Hence for spin eigenfunctions, the quantum mechanical transformation between matter and antimatter in Eq. 63 is essentially equivalent to the proposed classical spatial reflection operator in Eq. 61. Hence we may infer that the classical analogue of antimatter is simply a mirror image of matter.

Although different from the quantum mechanical interpretation, the classical interpretation of matter and antimatter being related by spatial reflection is consistent with the universal observation that "matter to the right is symmetrical with antimatter to the left." [37] In Wu's famous experiment on beta decay of Cobalt-60, classical physics would interpret the mirror-image process as one involving anti-Cobalt-60. [35, 38] Classically, there are no right-handed neutrinos because the mirror image of a left-handed neutrino is a right-handed anti-neutrino. [39] According to classical physics, "parity violation" is just the mundane observation that matter is more common than anti-matter. This is quite different from the conventional quantum mechanical interpretation of spatial reflection, in which the mirror image of matter is still presumed to be matter, and not antimatter.

Notice that the phase factor in front of the bispinor in Eq. 54 is half of the phase of the real vector potential $\mathbf{Q}$. This suggests a classical physics analogue of fermions and bosons: the real-valued vector wave functions have orbital angular quantum numbers (ℓ) that are odd for classical fermions and even for classical bosons. Since the parity of the spherical harmonics is $(-1)^\ell$, particles with odd values of ℓ have distinct mirror-images corresponding to antiparticles. Particles with even values are their own mirror images. In the Standard Model, elementary fermions (quarks and leptons) likewise have distinct antiparticles. Furthermore, nearly all bosons are considered to be their own antiparticle, with notable exceptions being the W^+ and W^- bosons. Classical physics would evidently regard these two particles as fermions.

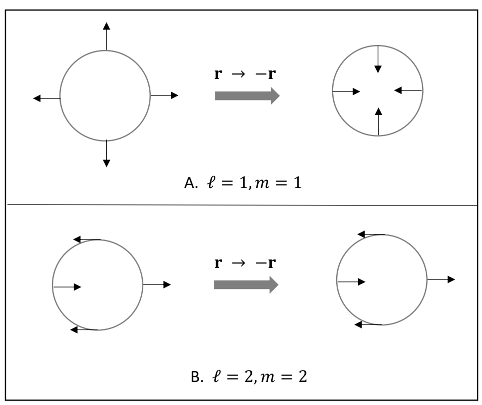

FIG. 1. Parity operator applied to spin density vector fields with (A) $\ell = 1, m = 1$, and (B) $\ell = 2, m = 2$.

3.3. Angular Eigenfunctions

Let $\Phi^{(+)}_{j,m_z}(\theta,\phi)$ and $\Phi^{(-)}_{j,m_z}(\theta,\phi)$ be the two-component spinor eigenfunctions of the angular momentum operators J^2, J_z, L^2, S^2, and:

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(+)}_{j,m_z}(\theta,\phi) &= (j-1/2)\Phi^{(+)}_{j,m_z}(\theta,\phi)\\ \boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(-)}_{j,m_z}(\theta,\phi) &= -(j+3/2)\Phi^{(-)}_{j,m_z}(\theta,\phi)\end{aligned} \tag{64}$$

where j and m_z are half-integer angular momentum quantum numbers. [40] These functions are related by $\sigma_r\Phi^{(+)}_{j,m_z} = \Phi^{(-)}_{j,m_z}$ and yield opposite eigenvalues under coordinate inversion ($\mathbf{r} \to -\mathbf{r}$). In terms of spherical harmonics $Y_{l,m_z}(\theta,\phi)$

they are:

$$\Phi^{(+)}_{j,m_z}(\theta,\phi) = \begin{bmatrix} \sqrt{\frac{j+m_z}{2j}} Y^{m_z-1/2}_{j-1/2}(\theta,\phi) \\ \sqrt{\frac{j-m_z}{2j}} Y^{m_z+1/2}_{j-1/2}(\theta,\phi) \end{bmatrix}$$

$$\Phi^{(-)}_{j,m_z}(\theta,\phi) = \begin{bmatrix} \sqrt{\frac{j+1-m_z}{2(j+1)}} Y^{m_z-1/2}_{j+1/2}(\theta,\phi) \\ -\sqrt{\frac{j+1+m_z}{2(j+1)}} Y^{m_z+1/2}_{j+1/2}(\theta,\phi) \end{bmatrix} \tag{65}$$

Consider a bispinor wave function with given j and m_z values of the form:

$$\psi(r,\theta,\phi) = \frac{1}{r}\begin{bmatrix} \tilde{i}g(r)\Phi^{(+)}_{j,m_z} \\ f(r)\Phi^{(-)}_{j,m_z} \end{bmatrix} \tag{66}$$

This functional form is commonly used in the Dirac representation (see e.g. Ref. 40), but here we are still using the chiral representation. Both representations yield the same spin density. The ambiguity stems from the fact that there is no oscillation, and therefore no distinction between wave propagation directions.

The spin density defined by such functions may be computed directly, and the curl of spin density is proportional to the classical velocity of the wave-carrying medium. For these wave functions with any values of j and m_z, the velocity field is purely azimuthal with the form $(0, 0, v_\phi(r,\theta))$ in spherical coordinates. For the simple case of $j = 1/2, m_z = 1/2$, the spin density is:

$$\mathbf{s} = (s_r, s_\theta, s_\phi) \propto ((1/r^2)(|f(r)|^2 + |g(r)|^2)\cos\theta, (1/r^2)(|f(r)|^2 - |g(r)|^2)\sin\theta, 0)\,. \tag{67}$$

The contributions from $\Phi^{(+)}_{1/2,1/2}$ and $\Phi^{(-)}_{1/2,1/2}$ can be determined by setting either $g(r)$ or $f(r)$ to zero, respectively. The $j = 1/2, m_z = 1/2$ wave function with $f(r) = 0$ is:

$$\psi^{(+)}(r,\theta,\phi) = \frac{\mathrm{i}\,g(r)}{\sqrt{4\pi}\,r}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{68}$$

The $j = 1/2, m_z = 1/2$ wave function with $g(r) = 0$ is:

$$\psi^{(-)}(r,\theta,\phi) = \frac{f(r)}{\sqrt{4\pi}\,r}\begin{bmatrix} 0 \\ 0 \\ \cos\theta \\ e^{\mathrm{i}\phi}\sin\theta \end{bmatrix} \tag{69}$$

The angular dependence of spin density for $(j, m_z) = (1/2, 1/2)$ is illustrated in Fig. 2. The bispinor orbital angular momentum numbers are $l = 0$ for $\Phi^{(+)}_{1/2,1/2}$ and $l = 1$ for $\Phi^{(-)}_{1/2,1/2}$. The vector fields have $l = 0$ and $l = 2$, respectively. The $l = 1$ vector field is not included in this formulation.

The velocity field computed from $\Phi^{(+)}_{1/2,1/2}$ by setting $f(r) = 0$ is $(v_r, v_\theta, v_\phi) \propto (0, 0, ((d/dr)|g|^2/r^2)\sin\theta)$. The velocity computed from $\Phi^{(-)}_{1/2,1/2}$ by setting $g(r) = 0$ is $(v_r, v_\theta, v_\phi) \propto (0, 0, ((2|f|^2 - r(d/dr)|f|^2)/r^3)\sin\theta)$.

While these computed velocity fields are consistent with spin densities similar to those found in quantum mechanics, they are not consistent with wave-like motion in an elastic solid. Such circulating motion cannot continue indefinitely. Time-dependence could be introduced explicitly (e.g. $f(r,t)$ and $g(r,t)$) while maintaining constant values of j and m_z, but such oscillations are not assumed in quantum mechanics. However, a constant component of azimuthal velocity is possible as part of a more general oscillatory motion. For example, if you could grab a point in the solid and move it in a circle around a central point, displacements would decrease with distance and the average velocity at all nearby points in space would be in the azimuthal direction. The "walking droplet" analogue of quantum mechanics also interprets the quantum wave function as an average or low-frequency part of a more general oscillation. [22]

3.4. Special Relativity

The mass term in quantum mechanics involves multiplication of the wave function by $\mathrm{i}\gamma^0$, which we have shown above to be a generator of rotation of wave velocity. This fact suggests that particles with mass should be interpreted

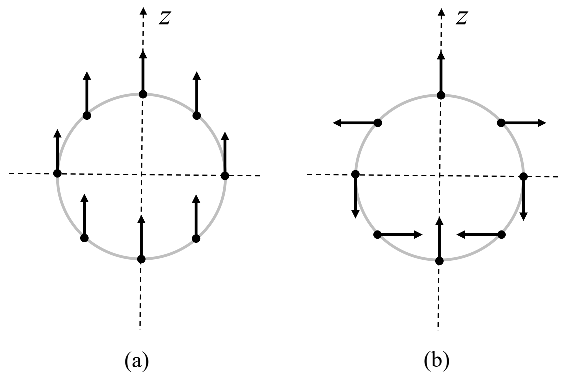

FIG. 2. Contribution to θ-dependence of spin density from (a) $\Phi^{(+)}_{1/2,1/2}$ and (b) $\Phi^{(-)}_{1/2,1/2}$

as waves whose velocity direction continuously rotates, or as standing waves consisting of a superposition of such waves. This behavior is similar to that of de Broglie waves in a central potential, whose rays follow circular paths between two bounding radii.[41] In the quantum mechanical interpretation of the Dirac equation, the fluctuation of position known as "zitterbewegung" is attributable to the particle undergoing circular motion with diameter equal to the Compton wavelength: $\lambda_0 = h/(m_0 c)$.[42]

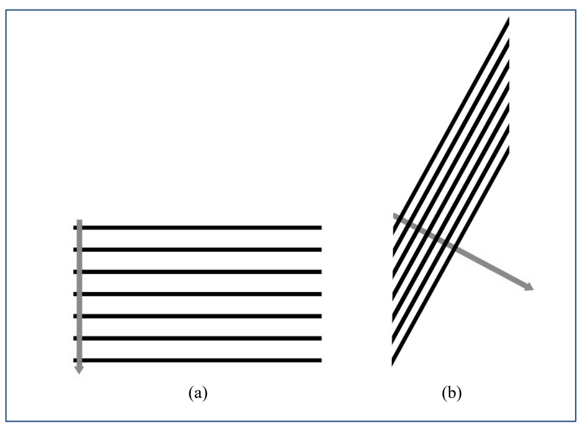

FIG. 3. (a) Model of circular wave propagation with the vertical axis representing the azimuthal direction. (b) Model of helical wave propagation with speed $v = 0.866c$ and $\gamma = 2$. These patterns are designed to be printed on a transparency sheet and rolled into a cylindrical tube (if printed on ordinary paper, shine a light into the tube in order to illuminate the wave pattern).

The model of particles as circulating waves offers a simple means for understanding special relativity (SR). [43–46] Construct a model as illustrated in Figs. 3 and 4. Black lines represent wave crests propagating at the speed of light. The arrows represent the distance light travels in one unit of time. When the sheet is rolled into a cylinder, the wave packet on the left is a stationary or standing wave with wave frequency $f_0 = m_0 c^2/h$ and wavelength $\lambda_0 = h/m_0 c$. The gray arrow represents the distance light travels in one unit of time, as measured by a stationary observer. The internal clock ticks once each time the wave traverses the circle.

Rotating the wave crests as in Fig. 3(b) results in helical wave propagation with average velocity $v = 0.866c$ and relativistic factor $\gamma = c/\sqrt{c^2 - v^2}$, with a new wavelength of λ_0/γ and relativistic frequency γf_0. The width of the moving wave packet is reduced by a factor of $1/\gamma$ (this length contraction was proposed by Fitzgerald and made quantitative by Lorentz in order to explain the null result of the Michelson-Morely experiment [47–49]). Propagation in the azimuthal direction, which measures time, is also reduced by a factor of $1/\gamma$ (time dilation). The distance between wave crests along the z-axis is $(\lambda_0/\gamma)(c/v) = h/(\gamma m_0 v) = h/p$, which is the de Broglie wavelength of a moving "particle". Hence the de Broglie wavelength results from a Lorentz boost of a stationary oscillation.

Consider the velocity triangle in Fig. 5 with hypotenuse c, one side representing average motion v, and a third side $\sqrt{c^2 - v^2}$ representing circulating motion perpendicular to the average motion. The Pythagorean theorem states

FIG. 4. Circulating wave model of an elementary particle rolled into a cylindrical tube.(a) Stationary particle wave packet. (b) Moving particle wave packet with $v = 0.866c$ and $\gamma = 2$.

that:

$$c^2 = v^2 + \left(\sqrt{c^2 - v^2}\right)^2 . \tag{70}$$

Simply multiply each side by $\gamma m_0 c$, with rest mass m_0, to obtain the energy-momentum-mass triangle. The Pythagorean theorem now yields:

$$(\gamma m_0 c^2)^2 = (\gamma m_0 c v)^2 + (m_0 c^2)^2 \tag{71}$$

which is equivalent to:

$$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{72}$$

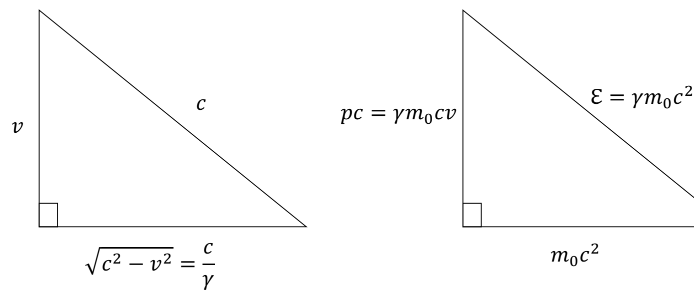

FIG. 5. Left: Velocity triangle with the lower side representing azimuthal propagation. Right: Energy-momentum triangle obtained from multiplication by $\gamma m_0 c$.

This relationship is valid, averaging over the cyclical motion, even if the average motion is in the plane of circulation. [44]

Since the wave equation is Lorentz-invariant and also arises for many different types of waves, SR should be understood as a general property of waves rather than a property of spacetime [43, 44]. A unifying principle of SR, applicable to all waves, is this:

> Measurements made by differently moving observers using a particular type of wave are related by Lorentz transformations based on the characteristic wave speed.

An explanation of this principle using animations is available online. [50]

Hence the model of the vacuum as an ideal elastic solid existing in a Galilean physical spacetime (with wave measurements comprising Minkowski spacetime) is entirely consistent with the laws of SR. The reader may recall that Maxwell also derived the equations of electromagnetism with the assumption of Galilean spacetime. Curiously, the success of Maxwell's model is sometimes regarded as evidence that his assumptions were wrong! In the words of Robert Laughlin, "Relativity actually says nothing about the existence or nonexistence of matter pervading the universe, only that any such matter must have relativistic symmetry. It turns out that such matter exists." [51] Einstein's postulate of the constancy of the speed of light may be understood as a recognition that all of our measurements are made using waves (including particle-like or standing waves) whose characteristic propagation speed is the speed of light. The current definition of the "meter" guarantees a constant measured speed of light (even though we know that the actual speed of light varies in a gravitational field). [52]

Many researchers have proposed that stationary elementary particles consist of standing waves or "solitons" rather than point-like singularities. [53–59] The model described above is a simplification of these more realistic models.

Interpretation of SR as a property of matter rather than spacetime clarifies the analysis of relative motion. Although it is impossible to measure absolute velocity, it is possible to measure absolute acceleration. If an inertial observer detects relativistic changes to accelerated clocks and rulers, it is certain that those changes are real, and they are consistent with the wave nature of matter. Acceleration changes matter, not the spacetime in which the matter moves. Likewise, an accelerated observer should realize that changes seen in external inertial clocks and rulers are not real, but are due to changes in the co-accelerated clocks and rulers used for comparison. Poincaré's statement that "we have no means of knowing whether it is the magnitude or the instrument that has changed" [60] does not apply to accelerated reference frames.

3.5. Lagrangian and Hamiltonian

Now we construct a Lagrange density $\mathcal{L}$. Lagrange's equation of motion for a field variable ψ is

$$\partial_t \frac{\partial \mathcal{L}}{\partial\left[\partial_t \psi\right]} + \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial\left[\partial_j \psi\right]} - \frac{\partial \mathcal{L}}{\partial \psi} = 0\,. \tag{73}$$

Multiplying Eq. 47 by $\mathrm{i}\psi^\dagger$ yields:

$$\psi^\dagger \mathrm{i}\partial_t \psi + c\psi^\dagger \gamma^5 \boldsymbol{\sigma}\cdot \mathrm{i}\nabla\psi - \psi^\dagger \chi \psi = 0, \tag{74}$$

Derivatives of $\psi^\dagger$ do not appear in this equation. Therefore we can construct a Lagrangian whose Euler-Lagrange equation has the simple form $\partial\mathcal{L}/\partial\psi^\dagger = 0$:

$$\mathcal{L} = \mathrm{i}\psi^\dagger \partial_t \psi + \psi^\dagger c\gamma^5 \boldsymbol{\sigma}\cdot \mathrm{i}\nabla\psi - \psi^\dagger \chi\psi\,. \tag{75}$$

The imaginary part of the Lagrangian has no physical significance, so we may discard it:[61]

$$\mathcal{L} = \mathrm{Re}\left\{\mathrm{i}\psi^\dagger \partial_t \psi + \psi^\dagger c\gamma^5 \boldsymbol{\sigma}\cdot \mathrm{i}\nabla\psi - \psi^\dagger \chi\psi\right\}\,. \tag{76}$$

From here on the representation of physical quantities as the real part of complex quantities will be implicit. The associated Hamiltonian is:

$$\mathcal{H} = p_\psi \partial_t \psi - \mathcal{L} = -\psi^\dagger c\gamma^5 \boldsymbol{\sigma}\cdot \mathrm{i}\nabla\psi + \psi^\dagger \chi\psi\,. \tag{77}$$

If we had kept nonlinear terms in Eq. 21, then the hamiltonian would contain addition terms such as $(1/2)\mathbf{w}\cdot\psi^\dagger(\boldsymbol{\sigma}/2)\psi$ whose volume integral equals kinetic energy.

The Hamiltonian operator defined by $\mathrm{i}\partial_t\psi = H\psi$ is: [8]

$$H\psi = -c\gamma^5 \boldsymbol{\sigma}\cdot \mathrm{i}\nabla\psi + \chi\psi\,. \tag{78}$$

In quantum mechanics, the hamiltonian represents energy density. We saw that for infinitesimal elastic plane waves, the quantities $\mathrm{Re}(\psi^\dagger \mathrm{i}\partial_t\psi)$ and $-\mathrm{Re}(\psi^\dagger \mathrm{i}c\gamma^5\boldsymbol{\sigma}\cdot\nabla\psi)$ are equal constants of the motion.

The Hamiltonian is a special case ($T^0_{\ 0}$) of the energy-momentum tensor:

$$T^\mu_{\ \nu} = \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\psi\right]}\partial_\nu\psi - \mathcal{L}\delta^\mu_{\ \nu}\,. \tag{79}$$

The conjugate momenta computed from the Lagrangian have the opposite sign of physical quantities. The dynamical (or wave) momentum density P_i is

$$P_i = -T^0_{\ i} = -\frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]}\partial_i\psi = -\psi^\dagger \mathrm{i}\partial_i\psi\,. \tag{80}$$

The wave angular momentum density is likewise

$$\begin{aligned}\mathbf{L} &= -\frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]}\partial_{\boldsymbol{\varphi}}\psi = -\mathrm{i}\psi^\dagger\partial_{\boldsymbol{\varphi}}\psi = -\mathrm{i}\psi^\dagger\frac{\partial r_i}{\partial\boldsymbol{\varphi}}\partial_i\psi \\ &= -\mathbf{r}\times\psi^\dagger \mathrm{i}\nabla\psi = \mathbf{r}\times\mathbf{P}\,.\end{aligned} \tag{81}$$

These dynamical variables are consistent with those of quantum mechanics. For total momentum ($\mathbf{P}+\mathbf{p}$) and angular momentum ($\mathbf{L}+\mathbf{s}$), we must combine the wave and medium contributions:

$$\mathbf{P}+\mathbf{p} = -\psi^\dagger \mathrm{i}\nabla\psi + \frac{1}{2}\nabla\times\psi^\dagger\frac{\boldsymbol{\sigma}}{2}\psi; \tag{82}$$

$$\mathbf{L}+\mathbf{s} = -\mathbf{r}\times\psi^\dagger \mathrm{i}\nabla\psi + \psi^\dagger\frac{\boldsymbol{\sigma}}{2}\psi\,. \tag{83}$$

The angular momentum operator is equivalent to that of quantum mechanics. The addition of intrinsic momentum to the wave momentum makes the energy-momentum tensor symmetric, as required for general relativity [62–64].

If the wave function is an eigenfunction of the spin component s_z with total spin $\hbar/2$, then the wave function should be normalized to $\int_V \psi^\dagger\psi dV = \hbar$. However, it is customary to normalize the wave function to unity, so all operators should be modified to include a factor of $\hbar$:

$$H\psi = -c\gamma^5\boldsymbol{\sigma}\cdot \mathrm{i}\hbar\nabla\psi + \hbar\chi\psi \tag{84a}$$

$$\mathbf{P}+\mathbf{p} = -\psi^\dagger \mathrm{i}\hbar\nabla\psi + \frac{1}{2}\nabla\times\psi^\dagger\hbar\frac{\boldsymbol{\sigma}}{2}\psi \tag{84b}$$

$$\mathbf{L}+\mathbf{s} = -\mathbf{r}\times\psi^\dagger \mathrm{i}\hbar\nabla\psi + \psi^\dagger\hbar\frac{\boldsymbol{\sigma}}{2}\psi\,. \tag{84c}$$

The normalization procedure amounts to regarding quantum mechanical dynamics as a special case of classical physics dynamics. There is no difference in the interpretation of dynamical quantities, so we conclude that spin angular momentum in quantum mechanics has the same interpretation as it has in classical mechanics: it is the angular momentum of the medium in which the waves propagate. Experimental confirmation of spin angular momentum is therefore evidence for the existence of an aether. Students should be encouraged to consider whether or not this evidence is convincing.

4. WAVE INTERACTIONS

Suppose we have two Dirac wave functions ψ_A and ψ_B, representing particle-like waves A and B. Adding the wave functions yields a total wave function ψ_T satisfying:

$$\begin{aligned}\psi_T^\dagger\boldsymbol{\sigma}\psi_T &= (\psi_A+\psi_B)^\dagger\boldsymbol{\sigma}(\psi_A+\psi_B)\\ &= \psi_A^\dagger\boldsymbol{\sigma}\psi_A + \psi_B^\dagger\boldsymbol{\sigma}\psi_B + \psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A\,.\end{aligned} \tag{85}$$

Since the spins must be additive, the total wave function is not generally the sum of the individual wave functions. However, we can treat the wave functions as being independent if the interference terms cancel [8]. This cancelation imposes a vector constraint on the wave functions:

$$\psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A = 0\,. \tag{86}$$

Assuming either of the waves to be a spin eigenfunction everywhere, one component of this constraint requires the wave functions to anti-commute:

$$\psi_A^\dagger\psi_B + \psi_B^\dagger\psi_A = 0\,. \tag{87}$$

For waves representing identical particles, this is the Pauli exclusion principle. This suggests that fermions are spin eigenfunctions.

The anti-commutation of wave functions is not true in general, but we can force the cancellation by introducing phase shifts at each point where the waves overlap. Such phase shifts have no effect on the actual dynamics of the total wave, but allow us to pretend that each particle wave maintains its separate identity even though there is actually only one combined wave. Hence superposition results in interaction potentials between two waves that are treated as "independent" particles. More study of such classical wave interactions is needed.

5. DISCUSSION

We have derived the fundamental equation of quantum mechanics, the Dirac equation, from a model of an ideal elastic solid. Others have also associated quantum mechanical behavior with waves in an elastic solid. [11, 65–68]

Unlike the non-relativistic Schödinger equation, the Dirac equation is fully relativistic and physically realistic. Each of the variables has a clear physical interpretation. In particular, spin angular momentum of elementary particles may be regarded as the angular momentum of the vacuum (or equivalently, the "aether"). While this interpretation might be contested, it is nonsensical to say that the aether is undetectable.

With finite motion, nonlinear terms would be added to the linear wave equation. Nonlinearity is a possible reason for quantized amplitudes. Many researchers have attempted to quantize the Dirac equation by adding nonlinear terms. [53–57, 69–72] Particle-like nonlinear wave solutions are sometimes called "breathers" or "solitons."

Thomas Jefferson famously wrote that "Ignorance is preferable to error; and he is less remote from the truth who believes nothing, than he who believes what is wrong" [73]. The non-relativistic Schrödinger equation is obviously wrong, and is therefore a poor choice for introducing students to the wave nature of matter. Students should first be taught the physical basis for the Dirac equation, after which the Schrödinger equation may be derived as an approximation in order to simplify the mathematics (see e.g. Refs. 74 and 75).

The model of the vacuum as an elastic solid also offers a good introduction to general relativity. Gravity, at least when weak, may be interpreted as ordinary refraction of waves toward regions whose wave speed is decreased by the presence of energy. [76–78] Wave speed in an elastic solid may likewise be decreased by stress-induced compression (increased inertial density). For example, twisting a rubber band induces a tension that tends to shorten it.

We showed how a model of stationary matter as standing waves gives rise to the de Broglie wavelength for moving particles. Recent research has revealed that classical physical systems can reproduce other quantum phenomena as well. In particular, silicone droplets bouncing on a vibrating water tank can exhibit single-particle diffraction and interference, wave-like probability distributions, tunneling, quantized orbits, and orbital level splitting.[16–22] Students (and their teachers) should be aware that many quantum behaviors have analogues in classical physics.

6. CONCLUSIONS

This paper offers a new approach for introducing students to the wave nature of matter, based on a classical wave description of incompressible motion in an elastic solid. Unlike the Schrödinger equation, this approach to wave mechanics is fully relativistic and includes spin angular momentum. Spin density is the field whose curl is equal to twice the incompressible momentum density. The second-order wave equation is transformed into a first-order Dirac equation, and sample plane wave solutions are given. The classical spatial reflection operator differs from that of the Standard Model. Odd and even angular quantum numbers for vector waves provide classical analogues of fermions and bosons. A model of stationary matter as circulating waves yields the relativistic energy-momentum equation for relativistic particles. A Lagrangian and Hamiltonian are constructed, from which the dynamical operators of relativistic quantum mechanics are derived. Interactions between waves that are spin eigenfunctions yields the Pauli exclusion principle and interaction potentials. Hence classical wave theory can be a powerful educational tool for modeling the wave properties of matter.

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