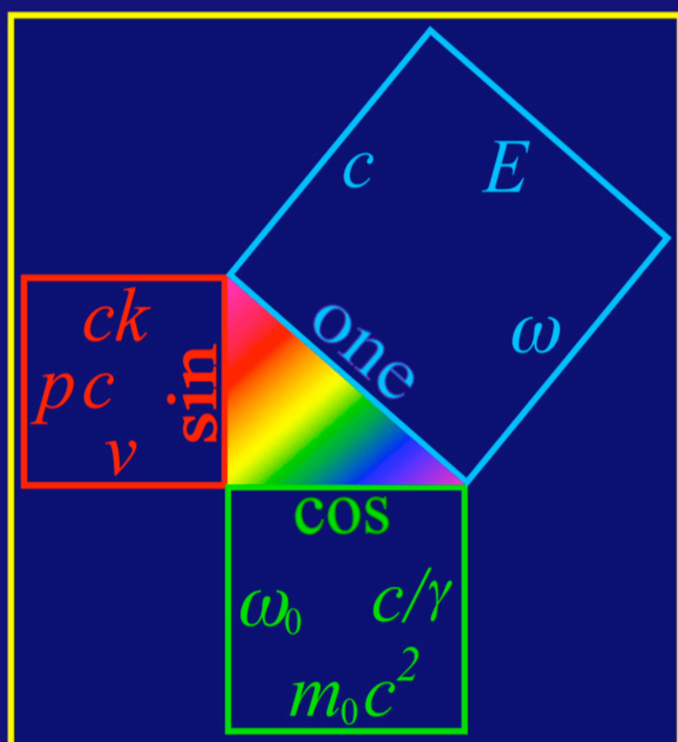
The Classical Wave Theory of Matter

A Dynamical Interpretation of Relativity, Quantum Mechanics, and Gravity



Robert A. Close

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2nd Draft

Robert A. Close, PhD

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Preface

> *"Reason and free enquiry are the only effectual agents against error."*
> – Thomas Jefferson, *Notes on the State of Virginia*, 1784

This book is the second draft of an attempt to bridge the conceptual gap between classical and modern physics. According to classical reasoning space is Euclidean, time is independent of position, light waves must travel through a material aether, and dynamical systems are deterministic. Modern physics asserts that space is curved, time is dependent on position and velocity, and fundamental physical processes are probabilistic. It is almost universally believed that experimental and theoretical developments of the 20th century not only disproved specific classical models, but in fact eliminated the possibility that any mechanistic model could properly describe nature. The title of this book will likely seem self-contradictory to physicists accustomed to a clear separation between classical and quantum physics. However, the reader will see that mechanistic models can in fact be used to explain fundamental physical phenomena that have hitherto been supposed to be beyond the realm of classical physics.

In accordance with common usage, I use the term 'classical physics' to refer to any mechanistic description of natural phenomena which presumes the existence of continuous media, Euclidean space, and absolute time. The title of this book refers to new applications of classical ideas and not to antiquated theories of the past.

I offer two arguments for consideration of mechanistic models of fundamental physical phenomena. The first reason is that in spite of past failures, it is still possible that a suitable mechanistic model can be found. Before classical physics can be rejected altogether, it must be proven that *all* mechanistic models yield predictions inconsistent with observations of nature. Since the number of possible mechanistic models is infinite, it is impossible to reject all of them unless they can be proven to share some incorrect feature. As the reader will see, features that have been presumed to fit this criterion have been incorrectly interpreted.

The second reason for studying mechanistic models of the universe is simply to build intuition. One can hardly expect to master the subtleties of matter waves or quantum fields in the mysterious vacuum without first being able to understand the behavior of undulations in a simple elastic solid. Yet this is precisely the limitation under which physicists have labored for the past century.

Historically, mechanistic models have figured prominently in the progress of physics [see e.g. Whittaker 1951]. The "rotationally elastic" aether introduced by James MacCullagh in 1839 provided a mechanistic model which was consistent with all of the known properties of light including polarization, reflection and refraction, and crystal-optics. William Thomson (Lord Kelvin) described how such a mechanical medium could be made. Joseph Boussinesq achieved similar success in 1867 by assuming an ordinary elastic solid aether that permeates matter as well as empty space. James Clerk Maxwell's historic formulation of the equations of electromagnetism in the early 1860's relied on a mechanical model of the aether consisting of elastic cells interspersed with rolling particles.

This last example is particularly significant in light of the rise of relativity theory at the beginning of the 20th century. The Lorentz transformations relating spatio-temporal coordinates of observers in relative motion are widely believed to be inconsistent with classical notions of space and time. Yet Maxwell's equations, which are covariant with respect to Lorentz transformations, were derived from a classical mechanical model. *How can a classical model of a mechanical aether be consistent with the Principle of Relativity?* This book answers this question by showing that the laws of Special Relativity are a consequence of the wave nature of matter.

With respect to quantum mechanics, virtually all students are introduced to the subject via study of the non-relativistic Schrödinger equation. This equation not only violates the principle of relativity, but it also reduces the four-component Dirac spinor of modern quantum theory to a single scalar variable. Simply put, the non-relativistic Schrödinger equation preserves some of the math but discards all of the physics. One should not expect that physical interpretations of Schrodinger's equation are applicable to real natural systems. Recent advances in classical physics have led some physicists to declare ”our present thinking about quantum mechanics is infested with the deepest misconceptions” [Gull et al 1993]. This book attempts to dispel such misconceptions by seeking clear, mechanistic explanations of natural phenomena. In this book the coupled equations describing an electron are derived from a simple model and interpreted quite naturally as a description of the propagation of classical waves carrying angular momentum. The 'wave-particle duality' of matter is explained by the fact that massive particles are soliton waves (localized oscillations).

This book is written at the level of a second- or third-year university physics course. It is assumed that the reader has already studied the basic properties of waves, is familiar with physical conservation laws, and has at least a basic understanding of wave analysis using Fourier transforms.

This book addresses several basic physics questions. Some examples are:

> "What rotational motion is associated with spin angular momentum?"
> "Can angular momentum be defined independently of the choice of origin?"
> "What is the dynamical interpretation of a spinor?"
> "Why are many aether models (including Maxwell's) consistent with Special Relativity?"
> "Why do matter and antimatter behave like mirror images of each other?"
> "Why is gravity so much weaker than other forces?"
> "How do waves propagate in an elastic solid?"

The term "matter" is used in the most general sense, including all manifestations of energy and not merely those with mass. Since photons can combine to form electrons and positrons, it is clear that massive and non-massive particles should be regarded as different modes of a single physical phenomenon. The term "inertial density" is used in place of "mass density" in the model of the vacuum since "mass" is a property of matter rather than a property of the vacuum itself.

Most of the historical information in this book has been gleaned from the excellent work of Sir Edmund Whittaker, *A History of the Theories of Aether and Electricity, Vols. I and II* (New York: Philosophical Library 1951 and 1954, respectively). The short synopses presented here are not intended to be complete in any sense. They simply provide a historical context from which certain questions about the nature of matter arose and were answered. Numerous significant contributions have necessarily been omitted for the sake of brevity.

Richard Feynman once remarked that if he could explain his work to the average person it wouldn't have been worth a Nobel Prize. However, Einstein's view was that "you do not really understand something unless you can explain it to your grandmother." This author agrees with Einstein, and I hope that this book will help physicists to understand nature in a manner that can be meaningfully shared with the rest of humanity.

– Robert Close

Portland, 2014

Glossary

$\mathbf{a}$	displacement vector
$\mathbf{A}$	magnetic vector potential
$\mathbf{B}$	magnetic field
e_{ij}	strain tensor
$\mathbf{E}$	electric field
ε	energy density
$\mathbf{F}$	force
$\boldsymbol{\phi}$	rotation angle
Φ	potential energy or electric potential
H	Hamiltonian
$\mathscr{H}$	Hamiltonian density
$\bar{\mathrm{i}}$	scalar unit imaginary
$\tilde{\mathrm{i}}$	pseudoscalar unit imaginary
$\mathbf{j} = \textsc{l} + \mathbf{s}$	angular momentum density (orbital + spin)
$\mathbf{J} = \mathbf{L} + \mathbf{S}$	angular momentum (orbital + spin)
$\boldsymbol{J}$	electrical current density
$\mathbf{k}$	wave vector
K	kinetic energy
$\textsc{l}$	orbital angular momentum density (associated with wave propagation)
$\mathbf{L}$	orbital wave angular momentum
L	Lagrangian
$\mathscr{L}$	Lagrangian density
m_0	rest mass
μ	elastic shear modulus
$\mathbf{p}$	wave momentum (density)
$\mathbf{P}$	momentum
$\mathbf{q} = \rho\mathbf{u}$	momentum density of medium
q	electric charge
$\mathbf{Q}$	angular potential
$\dot{\mathbf{Q}}$	spin angular momentum
$\mathbf{r}$	position vector
ρ	inertial density
$\mathbf{s}$	spin angular momentum density (associated with rotations of medium)
$\mathbf{S}$	spin angular momentum
$\boldsymbol{\tau}$	torque density
$\mathbf{T}$	torque
$\boldsymbol{\Theta}$	rotation angle
$\mathbf{u}$	medium velocity
U	potential energy

$\mathbf{v}$	particle or wave velocity
$\mathbf{w} = \partial\mathbf{\Theta}/\partial t = (\nabla\times\mathbf{u})/2$	vorticity (angular velocity) of medium
ω	angular frequency
$\boldsymbol{\omega}$	angular velocity
$\Omega_i = \partial\Theta_i/\partial x_i$	torsion component
$\nabla\cdot\mathbf{\Theta}$	scalar torsion
ξ	coordinate variable
ψ	Dirac bispinor wave function

There may be exceptions to the above definitions, but I have tried to keep them to a minimum.

References

Gull S, Lasenby A, and Doran C 1993 Imaginary Numbers are not Real - the Geometric Algebra of Spacetime

Whittaker E 1951 *A History of the Theories of Aether and Electricity*, vol. 1 (Edinburgh: Thomas Nelson and Sons Ltd.)

Whittaker E 1954 *A History of the Theories of Aether and Electricity*, vol. 2 (Edinburgh: Thomas Nelson and Sons Ltd.)

Chapter 1. Review of Classical Physics

If you would be a real seeker after truth, you must at least once in your life doubt, as far as possible, all things.
– René DesCartes, *Discours de la Méthode* (1637).

1.1. Basic Ideas

All of physics is either impossible or trivial. It is impossible until you understand it, and then it becomes trivial.
– Ernest Rutherford

Progress in science arises from attempts to conciliate observations and predictions. First, a phenomenon is described in detail on the basis of observations or measurements. Second, a set of scientific principles, or laws, is invented to explain the observations. This set of laws is called a theory. A scientific theory must be capable of yielding verifiable predictions (otherwise the theory is not scientific, though it might still be correct). Sometimes multiple theories yield the same predictions of observed phenomena, in which case the simplest theory is considered to be the best (Ockham's razor). It often happens that a theory predicts phenomena that have not yet been observed and described. This leads to renewed efforts of observation and description. If new observations are not completely explained, then the cycle of prediction and observation is repeated.

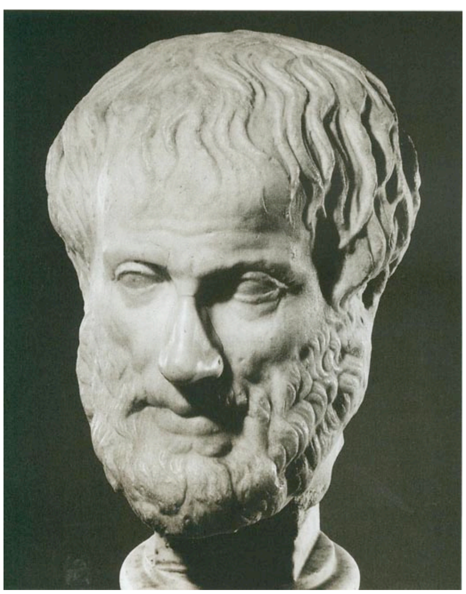

Figure 1.1 Aristotle (circa 384-322 BC)

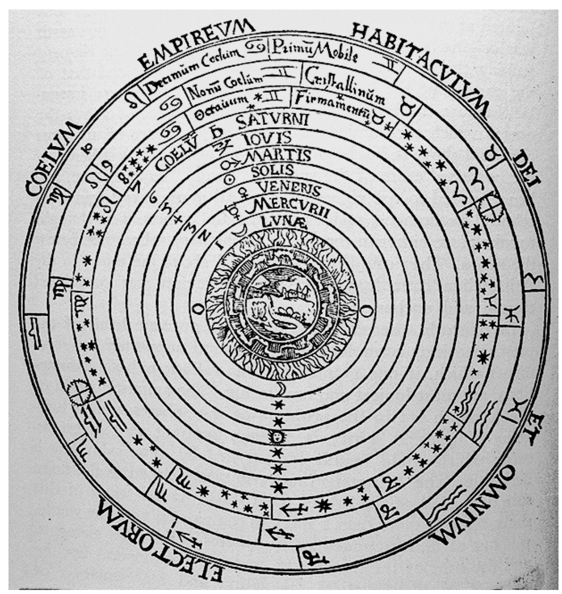

Figure 1.2 Aristotle's Universe

A classic example of this process is the development of the law of gravity on the basis of astronomical observations. The motion of stars and planets has been observed and studied since the dawn of human civilization. A mechanical basis for these motions was described by Aristotle [**Figure 1.1**] around 350 BC in his treatise "On the Heavens". Aristotle supposed the heavens to consist of concentric spheres into which celestial objects were affixed [**Figure 1.2**]. The outermost, or primary, sphere contained a multitude of stars. Other spheres contain only a single celestial object (sun, moon, or planet). The spherical earth was at the center (though the Pythagoreans believed otherwise). This model provides a good explanation of the motion of distant stars (which we now attribute to earth's rotation), but offers no physical explanation for the apparent irregular motion of planets with respect to the earth.

Figure 1.3 Ptolemy (AD 127-145)

An improvement on Aristotle's model of heavenly motion was reported by Claudius Ptolomaeus (Ptolemy) [**Figure 1.3**] in the second century A.D. Ptolemy's model [**Figure 1.4**] shifted the earth from the center of each orbit to a point called an eccentric, which was coupled with an equant point off-center in the opposite direction. Circular motion was attributed to rotation of a sphere, called a deferent. Non-circular motions were modeled by additional spheres, called epicycles, rotating about points on the deferent. Apollonius of Perga (Pergaeus) had proven that elliptical motion could be described in this way. Further corrections to this model were made by placing epicycles on epicycles. In principle, any periodic motion could be described by successive perturbations of this model. However, the complexity of this method reportedly led King Alfonso X of Spain to complain, "If the Lord Almighty had consulted me before embarking upon creation, I should have recommended something simpler."

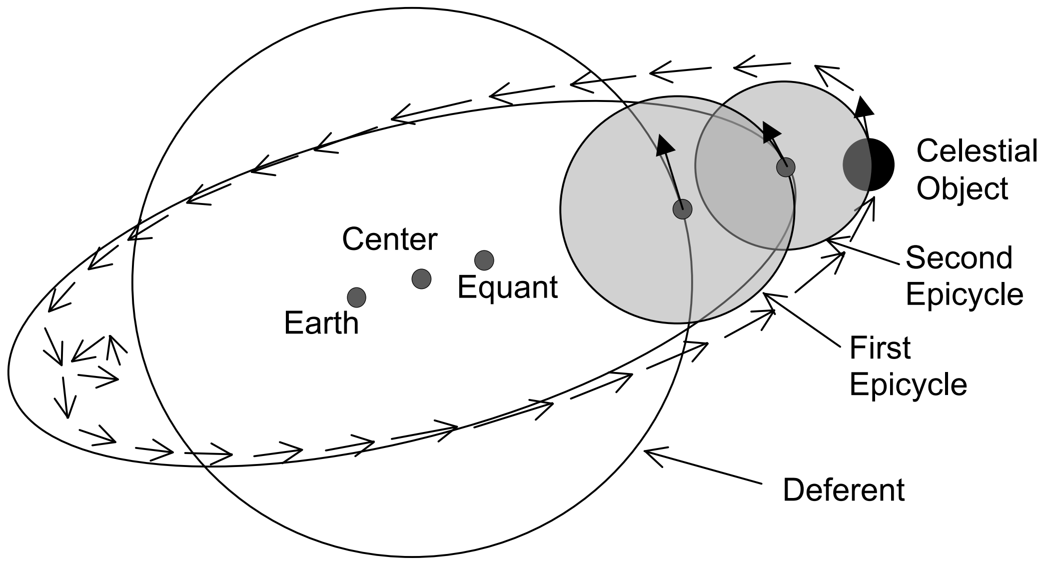

Figure 1.4 Example of elliptical (first epicycle) and retrograde (second epicycle) orbits produced using epicycles.

Nicolaus Copernicus [**Figure 1.5**] simplified this model in 1514 by reasoning that the sun, rather than the earth, is at the center of the rotating spheres [On the Revolutions of the Celestial Spheres]. This new theoretical development reduced the number of epicycles necessary to compute planetary motion. In particular, the apparent retrograde motion of planets was made consistent with regular circular orbits.

Figure 1.5 Nicolaus Copernicus (1473-1643)

In 1605, Johannes Kepler [**Figure 1.6**] eliminated the need for multiple epicycles by placing planets in elliptical orbits with the sun at one focus of the ellipse, rather than at the center. He also deduced that a line from the sun to a planet sweeps out equal areas in equal times, and that the square of the orbital period of a planet is proportional to the cube of the length of the semi-major axis.

Figure 1.6 Johannes Kepler (1571-1630)

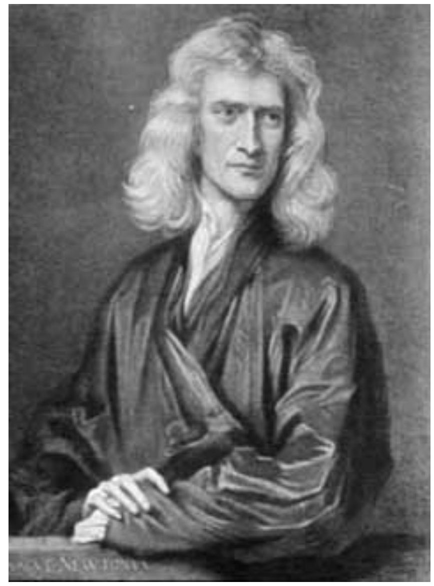

Figure 1.7 Isaac Newton (1643 - 1727)

Kepler's three laws of planetary motion greatly simplified the computation of planetary orbits. Nonetheless, Isaac Newton [**Figure 1.7**] found an even simpler principle to explain the orbits. Each planet is attracted to the sun by a gravitational force proportional to the inverse square of the distance from the sun. This gravitational force not only explained planetary motion, it also explained the attraction of terrestrial objects toward the earth. Yet as ever more accurate measurements were made, even this elegant theory did not explain all of the planetary motions. In particular, the perihelion of the orbit of Mercury advanced at a rate which could not be explained by the influence of other planets.

Albert Einstein's theory of general relativity finally explained these anomalies of planetary motion [**Figure 1.8**]. In this theory a gravitational field is proportional to a gradient in the speed of light (this property of general relativity is discussed in Chapter 4), which is reduced by the presence of energy. The theory also predicts that light waves from distant stars refract toward massive objects such as the sun or planets. This prediction has been verified by measuring, during solar eclipses, the positions of stars whose light propagates close to the sun on the way to earth. Near black holes, the refraction is so strong that the light cannot escape.

Figure 1.8 Albert Einstein (1879 – 1955)

Thus the complicated clockwork of Ptolemy's spheres has been replaced by the physical principle that gravity results from the presence of energy in space.

Modern physics now has its own version of Ptolemy's spheres in the realm of particle physics. An empirical method, quantum field theory, has been found to accurately compute statistical outcomes of experiments. The method works but it relies on myriad empirical constants and has defied explanation as to why nature should behave this way. Like Ptolemy's spheres with epicycles on epicycles, predictions of quantum field theory are computed using successive approximations or perturbations. Each correction is represented by a Feynman diagram, which is interpreted as representing an interaction of elementary particles [**Figure 1.9**]. A "renormalization" procedure adjusts interaction strengths in order to produce the correct final results.

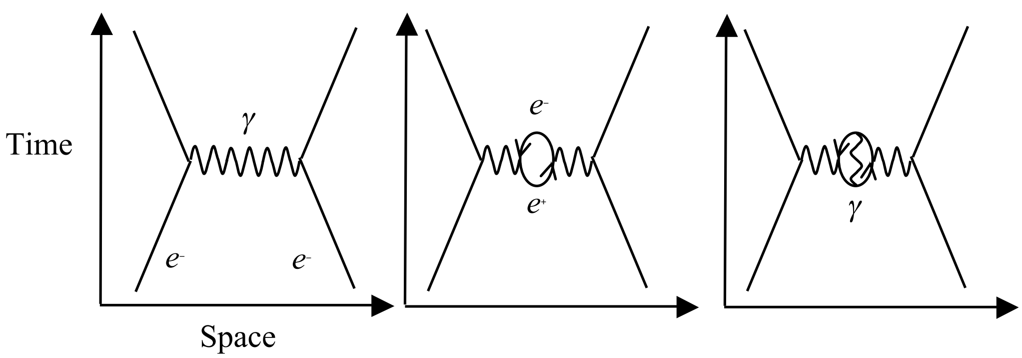

Figure 1.9 Renormalization in quantum electrodynamics: An interaction between two electrons (e^-) is modeled from left by (1) "virtual" photon (γ) exchange, (2) additional intermediate creation and destruction of electron-positron pair (e^-, e^+), and (3) additional virtual photon (γ) exchange. Additional processes contribute *ad infinitum* to the interaction.

In the case of Ptolemy's spheres, the difficulties arose because astronomers lacked an understanding of gravity and attempted to describe celestial orbits in terms of circular motion.

The method was sufficient at the time for its purpose of predicting future positions of celestial objects in the sky. Its major flaw was unnecessary complexity. Each planet had its own set of epicycle parameters to define the orbit. Newton's (and later Einstein's) theory of gravity unified all celestial orbits under a single physical principal dependent only on the masses of the objects in the sky.

In modern theories of elementary particles, each type of particle is assigned a unique set of parameters (mass, spin, isospin, electric charge, weak charge, strong charge, etc.). In fact, each particle has its own equation of evolution, which includes interactions with other particles. The catalog of elementary particles and their associated constants, in combination with the computational methods of quantum field theory, is called the Standard Model of particle physics. The Standard Model has proven to be extremely accurate in every case which has been tested. Its flaw is not inaccuracy but complexity. Equations for basic physical quantities such a angular momentum have no single representation but are dependent on which particles are present in space. Quantum numbers called "up", "down", "strangeness", "beauty", "charm", and "truth" have been invented to differentiate particles on the basis of unexplained properties. Simple physical questions such as "What rotation is associated with 'spin' angular momentum?" are left unanswered. No rationale has ever been accepted to explain why particles should exhibit the wave-like characteristics that are observed in experiments. And gravity has no obvious relation to the Standard Model.

Elementary particles, despite the name, are not immutable. For example, two photons can collide and transform into an electron-positron pair. The inverse transformation is also possible. Thus it is obvious that photons, electrons, and positrons are not truly "elementary" particles but instead represent different states of a shared physical process. Other particles undergo similar transformations of identities. Hence there is reason to believe that all elementary particles are different manifestations of a single physical process, and might be described as modes or perturbations of the vacuum.

The premise of this book is that the conceptual difficulties of quantum theory arise because physicists attempt to describe matter as discrete independent particles rather than as continuous waves. We will derive the basic dynamical and statistical properties of matter from a simple wave model, with massive particles interpreted as soliton waves. However, we will not derive the numerical predictions of the theory as this remains an unsolved problem.

Our topics will include special relativity, particle propagation, interactions, spin, statistics, and gravitational attraction. The reader is assumed to have prior understanding of wave processes at the undergraduate level. Readers of this book will become acquainted with the mathematics of quantum mechanics and relativity. They will then be free to further their understanding of modern physics either by further classical analysis or by more traditional avenues of study. We begin with a review of classical physics.

1.2. Classical mechanics

Truth is by nature self-evident. As soon as you remove the cobwebs of ignorance that surround it, it shines clear.
– Mohandas Gandhi

Classical mechanics typically begins with Newton's laws of motion:

(1) an object in motion moves in a straight line at constant speed unless a force acts upon it (momentum **p**=constant if force **F**=0)

(2) An object's momentum changes at a rate proportional to the force on the object ($\mathbf{F}=d\mathbf{p}/dt$)
(3) Any action on an object results in an equal and opposite reaction from the object. In other words, the force ($\mathbf{F}_B$) which object A exerts on B is equal and opposite to the force ($\mathbf{F}_A$) which B exerts on A.

Actually, the first and third 'laws' can be regarded as special cases of the second law. The first law simply describes the case of zero force: $\mathbf{F}=0$. The third law can be derived from the second simply by considering the combination of two objects A and B as a single 'object'. In the absence of external forces ($\mathbf{F}=0$) the first law requires that the total momentum is $\mathbf{p}_A + \mathbf{p}_B = \text{constant}$. Taking the time derivative yields:

$$\frac{d\mathbf{p}_A}{dt} + \frac{d\mathbf{p}_B}{dt} = 0 \tag{1-1}$$

This implies that any change in momentum of object A must be accompanied by an equal and opposite change in momentum of object B. The third law is obtained by substitution of forces for the rates of change of momenta.
Angular momentum $\mathbf{L}$ is defined as the cross product of the momentum with a displacement vector $\mathbf{r}$:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1-2}$$

The integral of force times displacement is called work:

$$W = \int \mathbf{F} \cdot d\mathbf{r} \tag{1-3}$$

This is the change in mechanical energy which results from application of a force. In differential form, this relation can be written as:

$$\nabla W = \mathbf{F} \tag{1-4}$$

In the case of a force applied to a solitary object, Newton's second law of motion yields:

$$\nabla W = \frac{d\mathbf{p}}{dt} \tag{1-5}$$

Note that the work performed in accelerating an object is energy which is transferred to the object. If we are discussing a reservoir of 'potential' energy U which is depleted to accelerate on object (or increased by deceleration), then we need a minus sign:

$$\nabla U = -\frac{d\mathbf{p}}{dt} \tag{1-6}$$

1.2.1. Conservation Laws

In a closed system, i.e. one for which we include all sources of force, the total force must be zero, and therefore the total momentum is constant. This law is known as conservation of momentum. A more abstract line of reasoning for momentum conservation goes like this: If an object (or group of objects) were to exert a net force on itself (themselves), then the energy of the object would depend on position. One would have to do work to move the object against the self-force. Conversely, if energy is independent of position then there is no self-force and momentum is conserved. With this line of reasoning, the law of momentum conservation follows from the assumption that energy is independent of position. This is an example of a physical symmetry.

Figure 1.10 Emmy Noether (1882-1935)

The mathematician Emmy Noether [**Figure 1.10**] proved that each symmetry in a physical system implies the existence of a conserved quantity [Noether 1918, Goldstein 1980]. The converse is also true. Some familiar symmetries and the corresponding conserved quantities are:

Symmetry	Conserved Quantity
translation	momentum
rotation	angular momentum
time shift	energy
spatial inversion	parity

The last of these may be new to some readers. Parity is the factor (P) which accompanies inversion of spatial coordinates ($\mathbf{x} \to -\mathbf{x}$). The coordinate variables themselves obviously have negative parity (P=−1). Momentum and velocity also have negative parity. Spatial inversion is equivalent to mirror imaging followed by a 180º rotation about the coordinate axis perpendicular to the mirror. Hence a top spinning clockwise has a spatially inverted image which also spins clockwise. Therefore angular momentum has positive parity (P=+1). More importantly, the equations governing angular momentum are not changed in the coordinate-inverted system (e.g. $\mathbf{L} = \mathbf{r} \times \mathbf{p}$). In other words, the equation for angular momentum is symmetric with respect to coordinate inversion. Parity conservation implies that the equations describing a physical system are symmetric (unchanged) under coordinate inversion. Since coordinate inversion exchanges left- and right-handedness, it could also be labeled 'handedness conjugation'.

Intuitively, what we see in a mirror seems just as physically realistic as what we see directly, so before the mid-1950's scientists generally assumed that parity is conserved in elementary physical processes (though not macroscopically since some molecules such as DNA are found exclusively in their right-handed form in nature). However, experiments have shown that certain physical processes are intrinsically left- or right-handed for certain particles. If the particles themselves (e.g. protons, neutrons, and electrons) are assumed to be their own mirror image, then these results imply parity violation in physical laws and not just in asymmetrical distributions of left- and right-handed objects.

Interestingly, all anti-matter particles behave exactly like mirror-images of matter. The simplest explanation of this mirror-symmetry is of course that anti-matter is the mirror-image of matter. Strangely, mainstream physicists have rejected this simple explanation because it is inconsistent with theoretical assumptions about how to calculate mirror images of quantum mechanical wave functions. We will address this issue in Chapter 3.

1.2.2. Spin

The term spin has two meanings in physics. The first meaning is the ordinary one, namely rotation about a local axis. If a rigid top, with center of mass $\mathbf{r}_0$ and intertial moment I, rotates with angular velocity $\boldsymbol{\omega}_s$ about its own axis and also moves with momentum $\mathbf{p}$ along a straight line, then the angular momentum of the spinning top is:

$$\mathbf{J} = \mathbf{r}_0 \times \mathbf{p} + I\boldsymbol{\omega}_s = \mathbf{L} + \mathbf{S} \tag{1-7}$$

The first term is sometimes called the 'orbital' angular momentum and the second term is called the spin angular momentum. However, this separation is somewhat artificial since the spin angular momentum for a top with uniform mass density ρ can also be written as:

$$\mathbf{S} = \int (\mathbf{r} - \mathbf{r}_0) \times \rho \mathbf{v}(\mathbf{r}) d^3\mathbf{r} \tag{1-8}$$

which has the same form as the orbital angular momentum. In quantum mechanics, the spin angular momentum is not explicitly related to rotational motion (this may be either a feature of nature or the result of our ignorance about nature; we will argue for the latter).

The second meaning of spin relates to the transformation of variable components under rotations. In this context the spin is the ratio of $\pi/2$ radians divided by the angle between independent states. A scalar field is described at each point in space by a single number independent of orientation. Therefore scalar fields have spin zero (infinite angle between independent states). A vector field is described at each point in space by three independent components with an angular separation of $\pi/2$ radians between any pair of independent components (e.g. coordinate axes). Therefore a vector field has spin one. A vector $\mathbf{A}$ transforms under local infinitesimal rotation $\delta\boldsymbol{\phi}$ as:

$$\delta\mathbf{A} = -\left(\delta\boldsymbol{\phi} \cdot \frac{\partial}{\partial\boldsymbol{\phi}}\right)\mathbf{A} \tag{1-9}$$

Note that rotation of **A** by $\delta\boldsymbol{\phi}$ is equivalent to rotation of the coordinate axes by $-\delta\boldsymbol{\phi}$. For example, the components transform under rotation of the *z*-axis as:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\phi_z & \sin\phi_z & 0 \\ -\sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1-10}$$

The differential form of this equation is:

$$\frac{\partial}{\partial\phi_z}\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}\Bigg|_{\phi_z=0} = -\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -A_y \\ A_x \\ 0 \end{pmatrix} \equiv -\sigma_z^{(1)}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1-11}$$

The minus sign is separated from the matrix by convention. The matrix $\sigma_z^{(1)}$ is called a spin matrix, and can be regarded as one of three components of a spin vector:

$$\sigma_z^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \sigma_y^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \sigma_x^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1-12}$$

The superscript (1) is used to distinguish these spin-1 matrices from other matrices introduced later. Using these matrices the differential change in the vector **A** undergoing arbitrary rotation is;

$$\delta\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = -\delta\boldsymbol{\phi}\cdot\frac{\partial}{\partial\boldsymbol{\phi}}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \delta\boldsymbol{\phi}\cdot\boldsymbol{\sigma}^{(1)}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \delta\phi_y A_z - \delta\phi_z A_y \\ \delta\phi_z A_x - \delta\phi_x A_z \\ \delta\phi_x A_y - \delta\phi_y A_x \end{pmatrix} \tag{1-13}$$

The final form simply lists the components of the curl operator:

$$\delta\mathbf{A} = \delta\boldsymbol{\phi}\times\mathbf{A} \tag{1-14}$$

The above analysis is for rotation about a local axis, *i.e.* in coordinates with **r**=0. For rotation about an arbitrary axis we must include the effects of the change in **r** due to rotation:

$$\delta\mathbf{A} = -\delta\mathbf{r}\cdot\nabla\mathbf{A} - \delta\boldsymbol{\phi}\cdot\frac{\partial}{\partial\boldsymbol{\phi}}\mathbf{A} = -(\delta\boldsymbol{\phi}\times\mathbf{r})\cdot\nabla\mathbf{A} + \delta\boldsymbol{\phi}\times\mathbf{A} = -\delta\boldsymbol{\phi}\cdot(\mathbf{r}\times\nabla)\mathbf{A} + (\delta\boldsymbol{\phi}\cdot\boldsymbol{\sigma})\mathbf{A} \tag{1-15}$$

In the final form above the first term is called the ‘orbital’ component and the second term the ‘spin’ component of the rotational transformation.

Of special interest is the case where the dynamical angular momentum **J** is itself expressed as a rotational transformation. We will see below that the dynamical angular momentum can be expressed as a derivative of a scalar (L) with respect to angular velocity:

$$\mathbf{J} = \frac{\partial L}{\partial \dot{\boldsymbol{\phi}}} \tag{1-16}$$

In this case there is no distinction between the two definitions of 'spin'. Although a scalar has spin of zero, we will see that nonzero 'spin' does arise when the function L is expressed in terms of other variables.

1.3. Variational Methods

An error does not become truth by reason of multiplied propagation, nor does truth become error because nobody sees it.
– Mohandas Gandhi

For simple systems it is easy to derive equations of motion simply by determining the forces and applying Newton's law ($\mathbf{F}=m\mathbf{a}$). However, in many situations such an analysis can be rather tedious. Instead, physicists have developed variational methods which utilize energy rather than force. The basic idea of variational methods is that while the true evolution of the system satisfies a certain equation, other fictitious evolutions can be parameterized by their differences, or errors, from the true equation. The correct equation of evolution can then be regarded as the one which minimizes these errors.

Consider, for example, a ball moving through the air under the influence of gravitational force:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{1-17}$$

In moving between arbitrary points A and B, the equation of motion can be integrated:

$$\int_A^B \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} = \int_A^B \mathbf{F} \cdot d\mathbf{l} = \Delta W_{AB} \tag{1-18}$$

where we have identified the right-hand side with the work done by the force acting on the ball. Rearranging yields:

$$\int_A^B \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} - \Delta W_{AB} = 0 \tag{1-19}$$

Writing the path length in terms of velocity ($d\mathbf{l} = \mathbf{v}dt$) yields:

$$\int_{t_A}^{t_B} \frac{d\mathbf{p}}{dt} \cdot \mathbf{v}dt - \Delta W_{AB} = 0 \tag{1-20}$$

For a ball of mass m, the momentum is $\mathbf{p} = m\mathbf{v}$ and the equation reduces to:

$$\frac{1}{2}\int_{t_A}^{t_B} m \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})dt - \Delta W_{AB} = 0 \tag{1-21}$$

The first term represents the change in kinetic energy K, so we have:

$$\Delta K_{AB} - \Delta W_{AB} = 0 \tag{1-22}$$

This equation expresses conservation of energy. Defining a potential energy U as the source of work ($U = U_0 - W$):

$$K + U = E = \text{constant} \tag{1-23}$$

where E represents the total energy.

The equation of motion results from minimization of the difference between kinetic and potential energy. Define the Lagrangian function as:

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m v^2 - U(\mathbf{r}) \tag{1-24}$$

We require that the path integral be stationary (zero first order change) with respect to variations of the Lagrangian along the path:

$$\delta \int_{t_1}^{t_2} L(\mathbf{r}, \mathbf{v}) dt = 0 \tag{1-25}$$

Use the first-order expansion:

$$\delta L(\mathbf{r}, \mathbf{v}) = \frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial v_i} \delta v_i = \frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial v_i} \frac{\partial}{\partial t} \delta x_i \tag{1-26}$$

Integration by parts yields:

$$\frac{\partial}{\partial t} \left[\frac{\partial L}{\partial v_i} \delta x_i \right] \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x_i} \delta x_i - \frac{\partial}{\partial t} \left[\frac{\partial L}{\partial v_i} \right] \delta x_i \right\} dt = 0 \tag{1-27}$$

The endpoints are assumed fixed, so the first term is zero. The remaining integral must be zero for arbitrary changes $\delta x_i(t)$. This condition yields the Euler-Lagrange equation:

$$\frac{\partial}{\partial t} \left[\frac{\partial L}{\partial v_i} \right] - \frac{\partial L}{\partial x_i} = 0 \tag{1-28}$$

The quantity $\partial L / \partial v_i = \partial L / \partial \dot{x}_i$ is called the conjugate momentum (or momentum conjugate to the coordinate x). For the Lagrangian in (**Error! Bookmark not defined.**), the conjugate momentum is:

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i} = m v_i \tag{1-29}$$

This is of course the usual definition of momentum.

The energy may be obtained from the Lagrangian using the procedure:

$$E = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m v^2 + U(\mathbf{r}) \tag{1-30}$$

Using momentum and position as the independent variables, the functional form of the energy is called the Hamiltonian:

$$H = K(\mathbf{r},\mathbf{p}) + U(\mathbf{r},\mathbf{p}) = \frac{p^2}{2m} + U(\mathbf{r}) \tag{1-31}$$

The equation of motion is then:

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} \tag{1-32}$$

which couples nicely with the equation for velocity:

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \tag{1-33}$$

The variables **r** and **p** in this example are called conjugate variables.

1.4. Wave Equations

Evidence is better than theory (論より証拠).
– Japanese proverb

Historically, matter has been thought of in terms of discrete particles. When we look at scene we instinctively segment it into discrete objects. Yet it is now clear that on a subatomic level the behavior of matter is governed by wave-like equations. In general, waves are generated by the perturbation of a continuous medium which has two properties: (1) resistance to change or inertia, and (2) reactivity to change or restoring force.

1.4.1. Elastic waves

For example, consider displacements of a small region in the interior of a solid. Because the region contains mass, it has inertia. We call the density of inertia ρ_m. Newton's first law of motion states that the region will not change its motion unless acted upon by an external force. Because elements of a solid are bound together by an elastic attraction, any stretching will result in restoring forces which oppose the stretching [Morse and Feshbach 1953].

We will use the shorthand notation ($\partial_t = \partial/\partial t; \quad \partial_i = \partial/\partial x_i$) to denote derivatives of field variables. The total derivative is:

$$\frac{d}{dt} = \partial_t + \mathbf{u}\cdot\nabla + \mathbf{w}\cdot\partial/\partial\boldsymbol{\varphi} \tag{1-34}$$

where **u** is the velocity and $\mathbf{w} = (\nabla\times\mathbf{u})/2$ is the angular velocity of the medium. The inertial reaction ($\mathbf{F}_\mathrm{I}$) to changes in the displacement **a** is given by Newton's second law:

$$\mathbf{F}_\mathrm{I} = \frac{d\mathbf{p}}{dt} = \rho_\mathrm{m}\left[\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + (\mathbf{w}\cdot\partial/\partial\boldsymbol{\varphi})\mathbf{u}\right]d^3r = \rho_\mathrm{m}\left[\partial_t^2\mathbf{a} + \dot{\mathbf{a}}\cdot\nabla\dot{\mathbf{a}}\right]d^3r \tag{1-35}$$

where we have used the differential mass $\rho_m d^3 r$. The restoring force is the result of stress (or tension) in the medium. A clear derivation of the relation between stress and strain can be found in *The Feynman Lectures on Physics, Vol. II* [R. P. Feynman, Robert B. Leighton, and Matthew Sands, (Addison-Wesley, Reading, 1963), Chapter 39]. We will give only a summary here. The strain in a medium represents the rate (spatial derivative) at which local displacements deviate from an equilibrium position. For example, if an object is stretched from equilibrium in the *x*-direction by an amount $a_x(x)$, the strain is given by:

$$e_{xx} = \frac{\partial a_x}{\partial x} \tag{1-36}$$

Note that if neighboring points are moved by the same distance (locally rigid displacement with $\partial u_x / \partial x = 0$) then there is no strain between those points. The first subscript indicates the direction of displacement. The second subscript indicates the direction of variation. If a region is rotated counterclockwise about the *z*-axis by an angle θ_z, the displacements are:

$$\begin{aligned} x' &= x\cos\theta_z - y\sin\theta_z \\ y' &= y\cos\theta_z + x\sin\theta_z \end{aligned} \tag{1-37}$$

The displacements can be written as:

$$\begin{aligned} a_x &= x' - x = x[\cos\theta_z - 1] - y\sin\theta_z \\ a_y &= y' - y = y[\cos\theta_z - 1] + x\sin\theta_z \end{aligned} \tag{1-38}$$

Notice that for finite rotations the displacement has a divergence ($\nabla \cdot \mathbf{a} \neq 0$) even though the motion is incompressible [**Figure 1.11**].

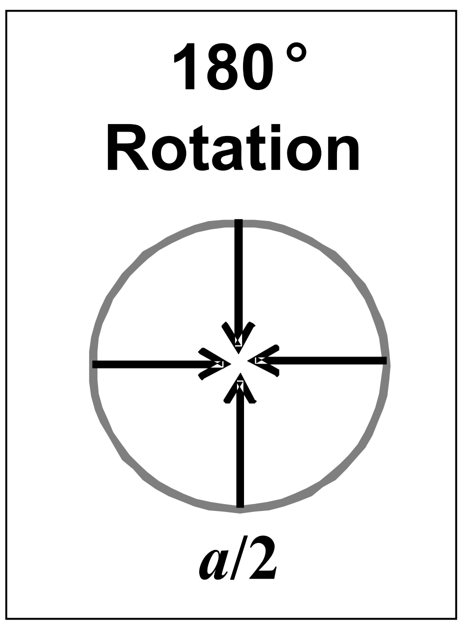

Figure 1.11 Diagram of half-displacements (*a*/2) for a 180° rotation, demonstrating non-zero divergence.

Incompressible motion requires only that the *velocity* have zero divergence. Since we are attempting to compute stress (including compression) in terms of displacements, we are forced to limit rotations to be infinitesimal in order to avoid anomalous divergences. For small rotations the first-order expression is:

$$a_x \approx -y\theta_z$$
$$a_y \approx x\theta_z \tag{1-39}$$

Of course, a pure rotation is rigid and does not introduce strain into the medium. This strain-free condition is satisfied if:

$$\frac{\partial a_y}{\partial x} + \frac{\partial a_x}{\partial y} = 0 \tag{1-40}$$

Shear strain comes about when the quantity above is not zero, and can be regarded physically as the deviation from rigid motion. The shear strain associated with rotations about the *z*-axis is defined as:

$$e_{xy} = e_{yx} = \frac{1}{2}\left[\frac{\partial a_y}{\partial x} + \frac{\partial a_x}{\partial y}\right] \tag{1-41}$$

Note again that this quantity is zero for pure infinitesimal rotations, but not zero in general for finite rotations. Incidentally, the corresponding component of the rotation is approximated by:

$$\omega_{xy} = -\omega_{yx} = \sin\theta_z = \frac{1}{2}\left[\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right] = \frac{1}{2}\left[\nabla \times \mathbf{a}\right]_z \tag{1-42}$$

We can compute the relative motion of neighboring points by combining the strain and rotation:

$$\frac{\partial a_i}{\partial x_j} = e_{ij} - \omega_{ij} \tag{1-43}$$

So far we have defined three components of the strain tensor: e_{xx}, e_{xy}, and e_{yx}. Other components of this tensor are easily obtained simply by using the appropriate indices.

The restoring forces which arise in response to strain are computed from a stress tensor (S_{ij}). The stress tensor contains the force per unit of oriented area which would result if a small block were cut out of the solid ($dA_i = \lfloor d\mathbf{x}_j \times d\mathbf{x}_k \rfloor \cdot \mathbf{n}_i$) but preserved its shape. For example, a positive value of S_{xz} means that the upper surface at (z+dz) would have a positive force in the *x*-direction while the lower surface at *z* would have a negative force in the *x*-direction. If the stresses are equal at both surfaces then there is no net force. And when the block is inside the solid the surface forces are cancelled by forces on the adjoining surfaces, which differ only by the opposite orientation of area. However, if the stress is non-uniform then a net force per unit volume is given by the derivative of stress along the direction of variation:

$$F_i = \sum_j \frac{\partial S_{ij}}{\partial x_j} \tag{1-44}$$

This expression is valid even when the block is placed back into the solid, since it is a force on the volume and not on the surface. Assuming a linear relation between stress and strain yields:

$$S_{ij} = \sum_{k,l} C_{ijkl} e_{kl} \tag{1-45}$$

The coefficients C_{ijkl} are called the tensor of elasticity. In an isotropic solid there can be no direction dependence in the elasticity coefficients, so any material-dependent coefficients must be scalars. The first is called the shear modulus (μ) and is conventionally multiplied by two to yield the elasticity coefficient:

$$S_{ij} \sim 2\mu e_{ij} \tag{1-46}$$

The second coefficient relates to compression. Compression is represented by $\nabla \cdot \mathbf{u} = \sum_k e_{kk}$

The resulting stress is proportional to the compression and equal in all directions:

$$S_{ij} \sim \lambda \left[\sum_k e_{kk} \right] \delta_{ij} \tag{1-47}$$

where the Kronecker delta is defined as $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij} = 0$ otherwise. The general expression for stress in an isotropic solid is therefore:

$$S_{ij} = 2\mu e_{ij} + \lambda \left[\sum_k e_{kk} \right] \delta_{ij} \tag{1-48}$$

The presence of stress does not necessarily imply a restoring force. After all, we can stretch a rubber band and hold it still so that the net force on any element of the band is zero. Restoring forces arise when the stress is non-uniform:

$$F_i = \sum_j \frac{\partial S_{ij}}{\partial x_j} \tag{1-49}$$

Plugging in the expressions for e_{ij} yields:

$$F_i = 2\mu \frac{\partial}{\partial x_j} \frac{1}{2} \left[\frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} \right] + \lambda \frac{\partial}{\partial x_i} \left[\nabla \cdot \mathbf{a} \right] \tag{1-50}$$

Or:

$$F_i = \mu \nabla^2 a_i + \left[\mu + \lambda \right] \frac{\partial}{\partial x_i} \left[\nabla \cdot \mathbf{a} \right] \tag{1-51}$$

Multiplication by fixed unit vectors yields the vector form:

$$\mathbf{F} = \mu\nabla^2\mathbf{a} + [\mu + \lambda]\nabla[\nabla\cdot\mathbf{a}] \tag{1-52}$$

We could replace the first term using the vector identity:

$$\nabla^2\mathbf{a} = \nabla[\nabla\cdot\mathbf{a}] - \nabla\times[\nabla\times\mathbf{a}] \tag{1-53}$$

to obtain:

$$\mathbf{F} = [2\mu + \lambda]\nabla[\nabla\cdot\mathbf{a}] - \mu\nabla\times[\nabla\times\mathbf{a}] \tag{1-54}$$

Note that in this form the forces separate into a term which depends on divergence ($\nabla\cdot\mathbf{a}$) and a term which depends on rotation $\left(\theta \approx (\nabla\times\mathbf{a})/2\right)$. This form reinforces the physical interpretation of shear strain as the deviation from rigid rotation.

Setting the restoring force equal to the inertial reaction yields the equation for displacement **a** in an elastic solid:

$$\rho_m\left[\frac{\partial^2\mathbf{a}}{\partial t^2} + \dot{\mathbf{a}}\cdot\nabla\dot{\mathbf{a}} + \frac{1}{2}[\nabla\times\dot{\mathbf{a}}]\cdot\frac{\partial}{\partial\boldsymbol{\varphi}}\dot{\mathbf{a}}\right] = [2\mu + \lambda]\nabla[\nabla\cdot\mathbf{a}] - \mu\nabla\times[\nabla\times\mathbf{a}] \tag{1-55}$$

The dot product of angular derivatives may be written as a curl ($[\partial/\partial\varphi_z]a_x = -a_y$, etc.):

$$\rho_m\left[\frac{\partial^2\mathbf{a}}{\partial t^2} + \dot{\mathbf{a}}\cdot\nabla\dot{\mathbf{a}} - \frac{1}{2}[\nabla\times\dot{\mathbf{a}}]\times\dot{\mathbf{a}}\right] = [2\mu + \lambda]\nabla[\nabla\cdot\mathbf{a}] - \mu\nabla\times[\nabla\times\mathbf{a}] \tag{1-56}$$

The convection and rotation terms are usually (though not correctly!) ignored:

$$\rho_m\frac{\partial^2\mathbf{a}}{\partial t^2} = [2\mu + \lambda]\nabla[\nabla\cdot\mathbf{a}] - \mu\nabla\times[\nabla\times\mathbf{a}] \tag{1-57}$$

We will also neglect convection and rotation for now, but will discuss them later.

1.4.2. Stress-energy tensor

The potential energy density is (Morse & Feshbach p.322):

$$\begin{aligned}U &= \frac{1}{2}\sum_{ij}S_{ij}e_{ij} = \frac{1}{2}\lambda\left[\sum_k e_{kk}\right]^2 + \frac{1}{2}\sum_{ij}\left[2\mu e_{ij}^2\right] \\ &= \frac{1}{2}\lambda[\nabla\cdot\mathbf{a}]^2 + \mu\left[\left(\frac{\partial a_x}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial y}\right)^2 + \left(\frac{\partial a_z}{\partial z}\right)^2\right] \\ &\quad + \frac{1}{2}\mu\left[\left(\frac{\partial a_x}{\partial y} + \frac{\partial a_y}{\partial x}\right)^2 + \left(\frac{\partial a_x}{\partial z} + \frac{\partial a_z}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial z} + \frac{\partial a_z}{\partial y}\right)^2\right]\end{aligned} \tag{1-58}$$

The Lagrangian for conventional elastic waves is given by:

$$L = K - U = \frac{1}{2}\rho\left(\frac{\partial \mathbf{a}}{\partial t}\right)^2 - \frac{1}{2}\sum_{ij} S_{ij}e_{ij} = \frac{1}{2}\rho\left(\frac{\partial \mathbf{a}}{\partial t}\right)^2 - \frac{1}{2}\sum_{ij}\left[2\mu e_{ij}^2\right] - \frac{1}{2}\lambda\left[\sum_k e_{kk}\right]^2 \tag{1-59}$$

Explicitly, this is:

$$L = \frac{1}{2}\rho\left(\frac{\partial \mathbf{a}}{\partial t}\right)^2 - \frac{1}{2}\lambda[\nabla\cdot\mathbf{a}]^2 - \mu\left[\left(\frac{\partial a_x}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial y}\right)^2 + \left(\frac{\partial a_z}{\partial z}\right)^2\right]$$
$$-\frac{1}{2}\mu\left[\left(\frac{\partial a_x}{\partial y} + \frac{\partial a_y}{\partial x}\right)^2 + \left(\frac{\partial a_x}{\partial z} + \frac{\partial a_z}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial z} + \frac{\partial a_z}{\partial y}\right)^2\right] \tag{1-60}$$

The canonical momentum represents the momentum density of the medium:

$$\mathbf{p}_i = \frac{\partial L}{\partial(\partial a_i/\partial t)} = \rho\frac{\partial a_i}{\partial t} \tag{1-61}$$

The stress-energy tensor (or momentum-energy tensor) is defined as:

$$W_{ij} = \sum_\alpha \frac{\partial a_\alpha}{\partial x_i}\frac{\partial L}{\partial(\partial a_\alpha/\partial x_j)} - L\delta_{ij} \tag{1-62}$$

The components may be written as:

$$W_{ij} = \begin{pmatrix} H & I_1 & I_2 & I_3 \\ -P_1 & W_{11} & W_{12} & W_{13} \\ -P_2 & W_{21} & W_{22} & W_{23} \\ -P_3 & W_{31} & W_{32} & W_{33} \end{pmatrix}$$

The Hamiltonian represents the total energy:

$$H = W_{00} = \sum_\alpha \frac{\partial a_\alpha}{\partial t}\frac{\partial L}{\partial(\partial a_\alpha/\partial t)} - L = K + U \tag{1-63}$$

The wave momentum density is given by:

$$P_i = -W_{i0} = -\sum_\alpha \frac{\partial a_\alpha}{\partial x_i}\frac{\partial L}{\partial(\partial a_\alpha/\partial t)} = -\sum_\alpha \frac{\partial a_\alpha}{\partial x_i}\rho\frac{\partial a_\alpha}{\partial t} \tag{1-64}$$

The minus sign is necessary (but sometimes ignored!) because a wave moving in the positive *x*-direction has time derivative opposite to the sign of the spatial derivative. Note that the direction of canonical momentum is determined only from the time derivative of displacement whereas the wave momentum also includes spatial derivatives. For shear waves the wave momentum is perpendicular to the direction of medium motion.

The wave intensity (**I**) also includes temporal and spatial derivatives, and represents a flow of energy:

$$I_i = W_{0i} = \sum_\alpha \frac{\partial a_\alpha}{\partial t} \frac{\partial L}{\partial(\partial a_\alpha / \partial x_i)} \tag{1-65}$$

The Euler-Lagrange equation is:

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial(\dot{a}_i)} = \frac{\partial L}{\partial(a_i)} \tag{1-66}$$

which yields:

$$\begin{aligned}
\int \rho\left(\frac{\partial^2 \mathbf{a}}{\partial t^2}\right) &= -\frac{\delta}{\delta \mathbf{a}} \int \left\{ \begin{aligned} &\frac{1}{2}\lambda[\nabla\cdot\mathbf{a}]^2 + \mu\left[\left(\frac{\partial a_x}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial y}\right)^2 + \left(\frac{\partial a_z}{\partial z}\right)^2\right] \\ &+\frac{1}{2}\mu\left[\left(\frac{\partial a_x}{\partial y} + \frac{\partial a_y}{\partial x}\right)^2 + \left(\frac{\partial a_x}{\partial z} + \frac{\partial a_z}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial z} + \frac{\partial a_z}{\partial y}\right)^2\right] \end{aligned} \right\} \\
&= (boundaryterms) + \int \left\{ \begin{aligned} &\lambda\nabla[\nabla\cdot\mathbf{a}] + 2\mu\left[\hat{\mathbf{x}}\left(\frac{\partial}{\partial x}\right)^2 a_x + \hat{\mathbf{y}}\left(\frac{\partial}{\partial y}\right)^2 a_y + \hat{\mathbf{z}}\left(\frac{\partial}{\partial z}\right)^2 a_z\right] \\ &+\mu\left[\begin{aligned} &\left(\hat{\mathbf{x}}\frac{\partial}{\partial y} + \hat{\mathbf{y}}\frac{\partial}{\partial x}\right)\left(\frac{\partial a_x}{\partial y} + \frac{\partial a_y}{\partial x}\right) + \left(\hat{\mathbf{x}}\frac{\partial}{\partial z} + \hat{\mathbf{z}}\frac{\partial}{\partial x}\right)\left(\frac{\partial a_x}{\partial z} + \frac{\partial a_z}{\partial x}\right) \\ &+\left(\hat{\mathbf{y}}\frac{\partial}{\partial z} + \hat{\mathbf{z}}\frac{\partial}{\partial y}\right)\left(\frac{\partial a_y}{\partial z} + \frac{\partial a_z}{\partial y}\right) \end{aligned} \right] \end{aligned} \right\}
\end{aligned} \tag{1-67}$$

This equation can be simplified by rearranging terms:

$$\begin{aligned}
&\int \rho\left(\frac{\partial^2 \mathbf{a}}{\partial t^2}\right) \\
&= \int \left\{ \begin{aligned} &\lambda\nabla[\nabla\cdot\mathbf{a}] + \mu\left[\hat{\mathbf{x}}\left(\frac{\partial}{\partial x}\right)^2 a_x + \hat{\mathbf{y}}\left(\frac{\partial}{\partial y}\right)^2 a_y + \hat{\mathbf{z}}\left(\frac{\partial}{\partial z}\right)^2 a_z\right] \\ &+\mu\left[\left(\nabla^2\mathbf{a}\right) + \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x}a_y + \frac{\partial}{\partial z}\frac{\partial}{\partial x}a_z\right) + \hat{\mathbf{y}}\left(\mathbf{x}\frac{\partial}{\partial z}\frac{\partial}{\partial y}a_z + \frac{\partial}{\partial x}\frac{\partial}{\partial y}a_x\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial z}a_x + \frac{\partial}{\partial y}\frac{\partial}{\partial z}a_y\right)\right] \end{aligned} \right\} \\
&= \int \left\{\lambda\nabla[\nabla\cdot\mathbf{a}] + \mu\nabla[\nabla\cdot\mathbf{a}] + \mu\left[\left(\nabla^2\mathbf{a}\right)\right]\right\}
\end{aligned} \tag{1-68}$$

Using the vector relation:

$$\nabla^2\mathbf{a} = \nabla[\nabla\cdot\mathbf{a}] - \nabla\times[\nabla\times\mathbf{a}] \tag{1-69}$$

yields the alternative form:

$$\int \rho\left(\frac{\partial^2 \mathbf{a}}{\partial t^2}\right) = \int\{[\lambda + 2\mu]\nabla[\nabla \cdot \mathbf{a}] - \mu\nabla \times [\nabla \times \mathbf{a}]\}$$

(1-70)

If we consider only shear waves:

$$\int \rho\left(\frac{\partial^2 \mathbf{a}}{\partial t^2}\right) = -\int\{\mu\nabla \times [\nabla \times \mathbf{a}]\}$$

(1-71)

It is interesting to note that this equation may also be derived from the Lagrangian:

$$L = \frac{1}{2}\rho\left(\frac{\partial \mathbf{a}}{\partial t}\right)^2 - \frac{1}{2}\mu\left[(\nabla \times \mathbf{a})^2\right]$$

(1-72)

As a simple example, consider transverse plane waves propagating in the *x*-direction with displacement in the *y*-direction. The Lagrangian is:

$$L = \frac{1}{2}\rho\left(\partial_t a_y\right)^2 - \frac{1}{2}\mu\left[\left(\partial_x a_y\right)^2\right] \tag{1-73}$$

The Hamiltonian represents the total energy:

$$H = W_{00} = \frac{1}{2}\rho\left(\partial_t a_y\right)^2 + \frac{1}{2}\mu\left[\left(\partial_x a_y\right)^2\right] \tag{1-74}$$

The wave momentum density is given by:

$$P_x = -W_{xt} = -\left(\partial_x a_y\right)\rho\left(\partial_t a_y\right) \tag{1-75}$$

Notice that a wave with opposite spatial and temporal derivatives has momentum in the positive *x*-direction.

The wave intensity (**I**) is:

$$I_x = W_{0x} = \left(\partial_t a_y\right)\mu\left(\partial_x a_y\right) = c_w^2 P_x \tag{1-76}$$

where $c_w = \sqrt{\mu/\rho}$ is the wave speed.

The Lagrangian for shear waves is essentially the form used by MacCullagh in 1837 to describe classical light waves. Empty space was modeled as a rotationally-elastic solid (called the aether). At the time, matter was thought to alter the density of the aether, so elimination of the compressional energy was thought necessary to prevent coupling to longitudinal waves at the interface. Later, Boussinesq proposed that the properties of aether were independent of the presence of matter, thereby allowing the aether to be regarded as an ordinary elastic solid.

This concludes the conventional analysis of shear waves in an elastic solid. Next we will derive an expression for angular momentum density.

1.4.3. Spin angular momentum density

Recall the conventional Lagrangian for shear waves:

$$L = \frac{1}{2}\rho(\partial_t \mathbf{a})^2 - \frac{1}{2}\mu\left[(\nabla\times\mathbf{a})^2\right] \tag{1-77}$$

The potential energy associated with shear waves can be interpreted as proportional to the square of the local rotation angle:

$$U = \frac{1}{2}\mu\left[(\nabla\times\mathbf{a})^2\right] \approx 2\mu\Theta^2 \tag{1-78}$$

This property suggests that shear waves may be described entirely by rotational variables. Consider a locally rigid rotation with velocity $\mathbf{v} = \mathbf{w}\times\mathbf{r} = -\mathbf{r}\times\mathbf{w}$, where $\mathbf{w}$ is the local angular velocity. This expression depends on the non-local quantity $\mathbf{r}$. However, the relation between velocity and angular velocity can be written in a local form as $d\mathbf{v} = -d\mathbf{r}\times\mathbf{w}$. For example, if $\mathbf{w}$ is in the z-direction then $dv_y = w_z dx$ and $dv_x = -w_z dy$. Therefore $w_z = dv_y/dx = -dv_x/dy = (\nabla\times\mathbf{v})_z/2$. Hence the differential equation corresponding to $\mathbf{v} = -\mathbf{r}\times\mathbf{w}$ is $\mathbf{w} = (\nabla\times\mathbf{v})/2$.

We desire a similar local spin angular momentum density whose curl is proportional to linear momentum. Based on the equation $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, we would expect the relationship to be $\rho\mathbf{u} = -(1/2)\nabla\times\mathbf{s}$. However, the angular momentum density must be the same sign as vorticity in order to have positive kinetic energy density $(1/2)\mathbf{w}\cdot\mathbf{s}$. It must also fall to zero at infinity in order to have finite total angular momentum. These conditions require an angular momentum density maximal at the axis of rotation and decreasing with increasing radius. This requires $d\mathbf{s} \sim -d\mathbf{r}\times\rho\mathbf{u}$, or:

$$\rho\mathbf{u} = +(1/2)\nabla\times\mathbf{s} \tag{1-79}$$

This sign is opposite to the expectation based on $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, but may be understood in light of the fact that conventional angular momentum density is zero at the axis and increases outward, whereas our spin angular momentum density starts at zero at infinity and builds up inward. Similarly, the time derivative of this relationship implies that force and torque densities are related by:

$$\mathbf{f} = +(1/2)\nabla\times\boldsymbol{\tau} \tag{1-80}$$

1.4.4. Rigid rotation

Consider a rigidly rotating cylinder of radius R. A consistent set of variables is:

$$\begin{aligned}
\mathbf{s} &= \hat{\mathbf{z}}\rho w_z\left[R^2 - r^2\right] \quad \text{to } r = R\text{, then zero}\\
\mathbf{u} &= \frac{1}{2\rho}\nabla\times\mathbf{s} = r w_z\hat{\boldsymbol{\varphi}} \quad \text{to } r = R\text{, then zero}\\
\mathbf{w} &= \frac{1}{2}\nabla\times\mathbf{u} = \hat{\mathbf{z}}\left[w_z - w_z R\delta(r-R)/2\right] \quad \text{to } r = R\text{, then zero}
\end{aligned} \tag{1-81}$$

Notice that if we had defined $\mathbf{s} = \mathbf{r}\times\rho\mathbf{u}$ then it would have z-dependent terms.

Since we are using cylindrical coordinates:

$$w_z = \frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r} r v_\varphi$$

The vorticity is defined so that the velocity drops to zero at $r=R$:

$$v_\varphi = \frac{1}{r}\int_0^r dr'\, 2r'\left[w_z - w_z R\delta\left(r'-R\right)/2\right] = \left\{ \begin{array}{cc} rw_z & \text{for } r < R \\ 0 & \text{for } r > R \end{array} \right\} \tag{1-82}$$

The total angular momentum per unit height is:

$$\begin{aligned} \mathbf{S} &= \int \mathbf{s}\, d^3 r = \int 2\pi r\, dr \rho w_z \left[R^2 - r^2\right] \\ &= \pi\rho w_z\left[R^4 - \frac{R^4}{2}\right] = \frac{\pi\rho w_z R^4}{2} = \frac{MR^2}{2} w_z = I w_z \end{aligned} \tag{1-83}$$

where I is the moment of inertia per unit height.

The kinetic energy in terms of angular momentum density is:

$$K = \frac{1}{2}\int \rho_m u^2 d^3 r = \frac{1}{8\rho_m}\int [\nabla\times\mathbf{s}]\cdot[\nabla\times\mathbf{s}] d^3 r \tag{1-84}$$

We can integrate by parts to express kinetic energy in terms of rotational variables, assuming that all derivatives are zero at infinity:

$$\begin{aligned} K &= \frac{1}{8\rho_m}\int [\nabla\times\mathbf{s}]\cdot[\nabla\times\mathbf{s}] d^3 r = \int\left[\partial_i s_j \partial_i s_j - \partial_i s_j \partial_j s_i\right] d^3 r \\ &= -\frac{1}{8\rho_m}\int\left[s_j \partial_i \partial_i s_j - s_j \partial_j \partial_i s_i\right] d^3 r = -\frac{1}{8\rho_m}\int\left[\mathbf{s}\cdot\nabla^2\mathbf{s} - \mathbf{s}\cdot\nabla[\nabla\cdot\mathbf{s}]\right] d^3 r \\ &= \frac{1}{8\rho_m}\int \mathbf{s}\cdot\left\{\nabla\times[\nabla\times\mathbf{s}]\right\} d^3 r = \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, d^3 r \end{aligned} \tag{1-85}$$

Notice that this result requires $\mathbf{w} = \left(1/4\rho_m\right)\nabla\times\nabla\times\mathbf{s}$, confirming our choice of sign in relating velocity $\mathbf{u}$ to angular momentum density $\mathbf{s}$.

For the rigidly rotating cylinder the total kinetic energy is:

$$\begin{aligned} K &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, d^3 r = \frac{1}{2}\int 2\pi r\, dr w_z \rho w_z \left[R^2 - r^2\right] \\ &= \pi\rho w_z^2\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi\rho w_z^2 R^4}{4} = \frac{MR^2}{4} w_z^2 = \frac{1}{2} I w_z^2 \end{aligned} \tag{1-86}$$

The delta function in vorticity does not contribute because the angular momentum density is zero there.

Notice that at a given point, kinetic energy density expressed in terms of vorticity is not equal to the kinetic energy density expressed in terms of velocity. In this sense any theory of spin angular momentum density is nonlocal. Spin angular momentum should be regarded as a potential that may be used to determine the local velocity.

1.4.5. Rotational waves

The elastic shear force equation:

$$\partial_t(\rho\mathbf{u}) = -\mu[\nabla\times(\nabla\times\mathbf{a})] = \mu\nabla^2\mathbf{a} \tag{1-87}$$

becomes:

$$\partial_t(\nabla\times\mathbf{s}) = 2\mu\nabla\times(\nabla\times\mathbf{a}) = 4\mu\nabla\times\mathbf{\Theta} \tag{1-88}$$

We can write the wave equation as:

$$\nabla\times\{\partial_t\mathbf{s}-\boldsymbol{\tau}\} = 0 \tag{1-89}$$

where the torque density $\boldsymbol{\tau} = -4\mu\mathbf{\Theta}$ is proportional to the local rotation.

Now define a variable $\mathbf{Q}$ whose Laplacian is equal to the rotation angle (or orientation):

$$\mathbf{\Theta} = -\frac{1}{4\rho}\nabla^2\mathbf{Q} \tag{1-90}$$

We assume that the medium has a linear response to rotation, so that the torque is:

$$\boldsymbol{\tau} = -4\mu\mathbf{\Theta} = c^2\nabla^2\mathbf{Q} \tag{1-91}$$

with $c^2 = \mu/\rho$.

Therefore, for infinitesimally small motion (neglecting convection $\mathbf{u}\cdot\nabla\dot{\mathbf{Q}}$), the variable $\mathbf{s}\equiv\dot{\mathbf{Q}}$ represents angular momentum density and we have:

$$\begin{aligned}
&\mathbf{s}\equiv\partial_t\mathbf{Q}\\
&\mathbf{u} = \frac{1}{2\rho}\nabla\times\mathbf{s} = \frac{1}{2\rho}\nabla\times\dot{\mathbf{Q}}\\
&\boldsymbol{\tau}\equiv c^2\nabla^2\mathbf{Q}\\
&\mathbf{w} = \frac{1}{2}\nabla\times\mathbf{u} = \frac{1}{4\rho}\nabla\times\left[\nabla\times\dot{\mathbf{Q}}\right]
\end{aligned} \tag{1-92}$$

We also have for infinitesimal displacements:

$$\begin{aligned}
&\mathbf{a}\approx\frac{1}{2\rho}\nabla\times\mathbf{Q}\\
&\mathbf{\Theta}\approx\frac{1}{2}\nabla\times\mathbf{a}\approx\frac{1}{4\rho}\nabla\times\left[\nabla\times\mathbf{Q}\right]\approx-\frac{1}{4\rho}\nabla^2\mathbf{Q}
\end{aligned} \tag{1-93}$$

Equating the rate of change of intrinsic angular momentum density to the torque density yields a wave equation for $\mathbf{Q}$:

$$\frac{\partial^2 \mathbf{Q}}{\partial t^2} = c^2 \nabla^2 \mathbf{Q} \tag{1-94}$$

For infinitesimal motions, the curl of this equation is the familiar shear elastic force equation (with $\nabla \cdot \mathbf{a} = 0$):

$$\nabla \times \left\{ \frac{\partial^2 \mathbf{Q}}{\partial t^2} - c^2 \nabla^2 \mathbf{Q} \right\} \approx \frac{\partial^2 \mathbf{a}}{\partial t^2} - c^2 \nabla^2 \mathbf{a} = 0 \tag{1-95}$$

This equation is more generally valid than the previous one. For example, a constant rigid rotation does not satisfy (1-94). However, (1-94) is sufficient for describing rotational waves.

For infinitesimal displacements, the Lagrangian density in angular variables is:

$$L' = \frac{1}{2} \dot{\boldsymbol{\theta}} \cdot \mathbf{s} - 2\mu\theta^2 \tag{1-96}$$

This is the usual difference between kinetic and potential energy. In this expression we must regard $\mathbf{s}$ as a function of $\dot{\boldsymbol{\theta}}$. Therefore the conjugate momentum to $\boldsymbol{\theta}$ is the angular momentum density:

$$\begin{aligned} p_{\Theta_j} &= \frac{\delta L}{\delta \dot{\Theta}_j} = \frac{1}{2} \frac{\delta}{\delta \dot{\Theta}_j} \int \dot{\boldsymbol{\Theta}} \cdot \mathbf{s} d^3 r = \frac{1}{8\rho} \frac{\delta}{\delta \dot{\Theta}_j} \int [\nabla \times \mathbf{s}] \cdot [\nabla \times \mathbf{s}] d^3 r \\ &= \frac{1}{8\rho} \int 2 [\nabla \times \mathbf{s}] \cdot \frac{\delta}{\delta \dot{\Theta}_j} [\nabla \times \mathbf{s}] d^3 r = \int \mathbf{s} \cdot \frac{\delta}{\delta \dot{\Theta}_j} \dot{\boldsymbol{\Theta}} d^3 r = \int \mathbf{s} \delta^3(\mathbf{r}) d^3 r = \mathbf{s}(\mathbf{r}) \end{aligned} \tag{1-97}$$

where $\delta^3(\mathbf{r})$ is the three dimensional Dirac delta function.
The Euler-Lagrange equation is:

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta(\dot{\theta}_i)} = \frac{\delta L}{\delta(\theta_i)} \tag{1-98}$$

which yields as above:

$$\frac{\partial}{\partial t} s_p = -4\mu\theta_p \tag{1-99}$$

As we saw above, the force equation corresponds to the curl of this equation. Although not completely general, this equation is valid for wave solutions.

Now we need to include the effects of finite amplitude. Finite velocity induces convection, and finite rotations can result in instantaneous rotations which are not parallel to the angular momentum. Adding these effects yields:

$$\partial_t^2 \mathbf{Q} = c^2 \nabla^2 \mathbf{Q} - \mathbf{u} \cdot \nabla \dot{\mathbf{Q}} + \mathbf{w} \times \dot{\mathbf{Q}} \tag{1-100}$$

If we assume $\mathbf{u} \cdot \nabla \dot{\mathbf{Q}} - \mathbf{w} \times \dot{\mathbf{Q}} = M^2 \mathbf{Q}$ then we obtain a transverse Klein-Gordon equation:

$$\partial_t^2 \mathbf{Q} = c^2 \nabla^2 \mathbf{Q} - M^2 \mathbf{Q} \tag{1-101}$$

This yields the relativistic energy-momentum relation between eigenvalues:

$$E^2 = c^2 p^2 + M^2 \tag{1-102}$$

Even if M=0, the convection and rotation terms may contribute significantly to the physical description of the wave.

We will discuss rotational waves further in Chapter 3.

1.4.6. Electromagnetism

The pinnacle of conventional physics was the development of a complete theory of electromagnetism in 1865 by James Clerk Maxwell [1891]. It was by then well understood that objects with electrical charge would attract if the charges were opposite and repel if the charges were the same sign. Accelerated charges resulted in the emission of electromagnetic waves which traveled at the same speed as light waves. Hence it was correctly deduced that light is a form of electromagnetic radiation. In modern notation Maxwell's equations can be written as:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho_e \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{J} \end{aligned} \tag{1-103}$$

where **E** is the electric field, **B** is the magnetic field, ρ_e is the electric charge density, and $\boldsymbol{J}$ is the electric current density.

The force exerted by the fields on a particle with charge q and moving with velocity **v** is given by the Lorentz force law:

$$\mathbf{F} = q\left[\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right] \tag{1-104}$$

For moving particles the current density is $\boldsymbol{J} = \rho_e \mathbf{v}$. The fields **E**, **B**, ρ_e, and $\boldsymbol{J}$ can be derived

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \\ 4\pi\rho_e &= -\nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} \\ \frac{4\pi}{c}\boldsymbol{J} &= \frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \times \left[\nabla \times \mathbf{A}\right] \end{aligned}$$

from potential fields **A** and Φ as follows:

(1-105)

Note that the density and current satisfy a continuity equation:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \qquad (1\text{-}106)$$

$$\Phi = \nabla \cdot \mathbf{G}$$

$$\mathbf{A} = -\frac{1}{c}\frac{\partial \mathbf{G}}{\partial t}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\Phi + \nabla \cdot \mathbf{A} = 0$$

$$4\pi\rho_e = -\nabla^2[\nabla \cdot \mathbf{G}] + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\nabla \cdot \mathbf{G} = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\Phi$$

$$\frac{4\pi}{c}\boldsymbol{J} = -\frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial^2 \mathbf{G}}{\partial t^2} - \nabla^2 \mathbf{G}\right] = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\mathbf{A}$$

The potentials in turn can be derived from a ‘super-potential’ **G**:

(1-107)

This set of potentials is called the Lorentz guage.

Alternatively we could pick out one direction $\hat{\mathbf{s}}$ from a divergence-free field which obeys a wave equation:

$$\Phi = \frac{1}{c}\frac{\partial}{\partial t}G_s$$

$$\mathbf{A} = \frac{\partial}{\partial x_s}\mathbf{G}$$

$$\nabla \cdot \mathbf{G} = 0$$

$$4\pi\rho_e = -\frac{1}{c}\frac{\partial}{\partial t}\nabla^2 G_s = -\nabla^2\Phi$$

$$\frac{4\pi}{c}\boldsymbol{J} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\nabla G_s + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_s}\mathbf{G} - \nabla^2\frac{\partial}{\partial x_s}\mathbf{G} = \frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi + \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\mathbf{A}$$

(1-108)

This set of potentials is called the Coulomb gauge. Since electrons (and protons) do have anisotropy due to spin, this definition of potentials is not as contrived as it might appear. Consider the case of electromagnetic fields in vacuum with ρ_e and $\boldsymbol{J}$ both zero. If we take the curl of the third of Maxwell's equations and combine it with the time derivative of the fourth equation we obtain:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \qquad (1\text{-}109)$$

Each of these equations is a homogeneous vector wave equation. In vacuum, both **E** and **B** have zero divergence, so these equations have the same form as the conventional equation for shear waves.

1.4.7. Wave Energy Density

The rate of work performed by electromagnetic fields interacting with charged particles in a volume V is:

$$\frac{\partial W}{\partial t} = \int_V d^3r\, \boldsymbol{J}\cdot\mathbf{E} \qquad (1\text{-}110)$$

By substituting for **J** and using some vector identities we can obtain (see Jackson p.236):

$$\frac{\partial W}{\partial t} = -\frac{1}{4\pi}\int_V d^3r\left[\frac{1}{2}\frac{\partial}{\partial t}\left[E^2 + B^2\right]\right] - \frac{c}{4\pi}\oint_S dS\, \mathbf{E}\times\mathbf{B} \qquad (1\text{-}111)$$

Clearly the first term on the right-hand side represents the energy associated with the fields in the volume while the second term represents the flux of energy through the surface of the volume. Therefore the energy associates with an electromagnetic field in volume V is:

$$U = \int_V d^3r\left[\frac{1}{8\pi}\left[E^2 + B^2\right]\right] \qquad (1\text{-}112)$$

This equation can be written in terms of potentials as:

$$U = \frac{1}{8\pi}\int_V d^3r\left[\left|-\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right|^2 + \left|\nabla\times\mathbf{A}\right|^2\right] \qquad (1\text{-}113)$$

1.4.8. Harmonic Representation of Waves

The following discussion is based on Jackson [1975 pp. 299-301].

The plane wave solution of the one-dimensional wave equation for a scalar A is:

$$A = A_{k,\omega}e^{i(kz-\omega t)} \qquad (1\text{-}114)$$

Both k and ω cannot be regarded as independent variables since the wave speed is given by $c = \omega / k$. Using $\omega = \omega(k)$, a general solution can be formed by integrating over all possible values of k:

$$A(x,t) = \frac{1}{\sqrt{2\pi}} \int dk \; A(k) e^{i[kz - \omega(k)t]} \tag{1-115}$$

The normalization factor is arbitrary but conforms to standard Fourier analysis.

For harmonic waves with the exponential ($\exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$), the derivatives transform as $\partial_t \to -i\omega$ and $\nabla \to i\mathbf{k}$. So the energy density of electromagnetic waves has the dependence:

$$U \sim \left| -i\mathbf{k}\Phi + i\frac{\omega}{c}\mathbf{A} \right|^2 + \left| i\mathbf{k} \times \mathbf{A} \right|^2 . \tag{1-116}$$

Since ω=ck we have

$$U \sim k^2 \left[\left| \hat{\mathbf{k}}\Phi + \mathbf{A} \right|^2 + \left| \hat{\mathbf{k}} \times \mathbf{A} \right|^2 \right] \sim \omega^2 \left[\left| \hat{\mathbf{k}}\Phi + \mathbf{A} \right|^2 + \left| \hat{\mathbf{k}} \times \mathbf{A} \right|^2 \right] \tag{1-117}$$

In the case of shear waves, the energy density is proportional to the stress times the shear ($dU = \mathbf{F} \cdot d\mathbf{x}$). We won't derive this explicitly, but to show that it is reasonable consider the following proportionalities:

$$U \sim \int \mathbf{f} \cdot d\mathbf{x} \sim \int \sum_{i,j} \frac{\partial S_{ij}}{\partial x_j} du_i \sim \int \sum_{i,j} dS_{ij} \frac{\partial u_i}{\partial x_j} \sim \int \sum_{i,j} dS_{ij} e_{ij} \sim S_{ij} e_{ij} \tag{1-118}$$

And since $S_{ij} \sim e_{ij} \sim \dfrac{\partial u_i}{\partial x_j}$ we conclude that harmonic waves have

$$U \sim \left| \partial u_i / \partial x_j \right|^2 \sim k^2 u^2 \sim \omega^2 u^2 \tag{1-119}$$

So for both electromagnetic waves and shear waves the energy density is proportional to the square of the frequency (or wave number). Of course, if one takes the derivative of the wave amplitude to be the true amplitude (e.g. using **E** and **B** instead of Φ and **A**), then the energy density becomes independent of ω and k. However, there is no conventional wave for which the energy density could be written so that it is proportional to an odd power of ω or k.

1.4.9. Separation of the Wave Equation

In many physical problems solutions of the wave equation can be found by the method of separation of variables. For example, waves inside a rectangular enclosure might be assumed to have the form:

$$\psi(x,y,z,t) = A(x)B(y)C(z)D(t)$$

The wave equation is then:

$$ABC\left[\frac{\partial^2}{\partial t^2}D\right]-\left[\frac{\partial^2}{\partial x^2}A\right]BCD-A\left[\frac{\partial^2}{\partial y^2}B\right]CD-AB\left[\frac{\partial^2}{\partial y^2}C\right]D=0 \tag{1-120}$$

Division by *ABCD* then yields:

$$\frac{1}{D}\frac{\partial^2}{\partial t^2}D-\frac{1}{A}\frac{\partial^2}{\partial x^2}A-\frac{1}{B}\frac{\partial^2}{\partial y^2}B-\frac{1}{C}\frac{\partial^2}{\partial y^2}C=0 \tag{1-121}$$

Each term in this equation is a function of only one variable. Therefore each term must be a constant, and the sum of the constants is zero. For example:

$$\frac{1}{c^2}\frac{1}{D}\frac{\partial^2}{\partial t^2}D=-\frac{\omega^2}{c^2}$$

$$-\frac{1}{A}\frac{\partial^2}{\partial x^2}A=k_x^2$$

$$-\frac{1}{B}\frac{\partial^2}{\partial y^2}B=k_y^2$$

$$-\frac{1}{C}\frac{\partial^2}{\partial z^2}C=k_z^2$$

$$-\frac{\omega^2}{c^2}+k_x^2+k_y^2+k_z^2=0 \tag{1-122}$$

Any function with $\omega^2=c^2k^2$ and of the form:

$$\psi(x,y,z,t)=e^{\pm i(\omega t-\mathbf{k}\cdot\mathbf{x})} \tag{1-123}$$

is clearly a solution. The general solution is a sum of these basis functions.

The wave equation is also separable in spherical coordinates (the following is based on Butkov [1968]). Assuming solutions of the form $A(r)B(\theta)C(\phi)D(t)$ we have:

$$\frac{1}{Dc^2}\frac{\partial^2 D}{\partial t^2}-\frac{1}{Ar^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}A\right]-\frac{1}{Br^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}B\right]-\frac{1}{Cr^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}C=0 \tag{1-124}$$

This yields the following separate equations:

$$\frac{1}{Dc^2}\frac{\partial^2 D}{\partial t^2} = -\frac{\omega^2}{c^2}$$

$$-\frac{1}{C}\frac{\partial^2}{\partial \phi^2}C = m^2$$

$$-\frac{1}{B\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}B\right] + \frac{m^2}{\sin^2\theta} = \lambda_\theta$$

$$-\frac{1}{Ar^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}A\right] + \frac{\lambda_\theta}{r^2} - \frac{\omega^2}{c^2} = 0 \tag{1-125}$$

Note that the polar solution $B(\theta)$ depends on the value of the azimuthal separation constant m^2, and the radial solution $A(r)$ depends on the values of the polar (λ_θ) and temporal $\left(\omega^2/c^2\right)$ separation constants.
The temporal solutions have the form:

$$D(t) = D_0 e^{\pm i\omega t} \tag{1-126}$$

The azimuthal solutions have the same form:

$$C(\phi) = C_0 e^{\pm im\phi} \tag{1-127}$$

Due to the periodic nature of the variable ϕ, the value of m is quantized. If m is an integer then $C(\phi)$ is single-valued ($C(2\pi)=C(0)$). Half-integer values of m allow for double-valued functions ($C(2\pi)=-C(0)$). Note that double valued functions $C(\phi)$ are single-valued for $|C(\phi)|2$.

Next we consider the polar equation. Multiplication by $B\sin\theta$ yields:

$$\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}B\right] + \lambda_\theta B\sin\theta - \frac{m^2}{\sin\theta}B = 0 \tag{1-128}$$

Letting $x = \cos\theta$ and $d\theta = dx/\sin\theta = dx/\left[1-x^2\right]^{1/2}$, we obtain the Associated Legendre Equation:

$$\frac{\partial}{\partial x}\left[\left[1-x^2\right]\frac{\partial}{\partial x}b(x)\right] + \lambda_\theta b(x) - \frac{m^2}{\left[1-x^2\right]}b(x) = 0 \tag{1-129}$$

where $b(x) = B(\theta(x))$. Note that $x=\pm1$ are singular points. We require that the solutions be finite at these points. Letting $b(x) = \left[1-x^2\right]^s u(x)$ yields the indicial equation:

$$-2s\left[1-x^2\right]u + 4x^2s^2u - 2xs\left[1-x^2\right]\frac{\partial}{\partial x}u - 2[s+1]x\left[1-x^2\right]\frac{\partial}{\partial x}u + \left[1-x^2\right]^2\frac{\partial^2}{\partial x^2}u \tag{1-130}$$
$$+\lambda_\theta\left[1-x^2\right]u(x) - m^2u(x) = 0$$

At $x=\pm1$ we have:

$$4s^2 - m^2 = 0 \tag{1-131}$$

which implies that $s=\pm m/2$. Since our purpose here is to make the solutions finite, we choose the positive value and assume $m>0$. The equation for $u(x)$ is now:

$$\left[1-x^2\right]\frac{\partial^2 u}{\partial x^2} - 2[m+1]x\frac{\partial}{\partial x}u - \left[m + m^2 + \lambda_\theta\right]u = 0 \tag{1-132}$$

We write $u(x)$ as a Frobenius series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1-133}$$

which yields the equation:

$$n[n-1]\sum_{n=0}^{\infty} a_n x^{n-2} - n[n-1]\sum_{n=0}^{\infty} a_n x^n - 2[m+1]n\sum_{n=0}^{\infty} a_n x^n - \left[m + m^2 + \lambda_\theta\right]\sum_{n=0}^{\infty} a_n x^n = 0 \tag{1-134}$$

Terms for each power of x must add to zero. This leads to the recurrence relation:

$$a_{n+2} = \frac{n[n-1] + 2[m+1]n + m + m^2 + \lambda_\theta}{[n+2][n+1]} a_n \tag{1-135}$$

Note that as $n \to \infty$ the recurrence relations becomes $a_{n+2} \to a_n$. For $|x|<1$ the power series converges since $x^n \to 0$ as $n \to \infty$. However, at $|x|=1$ the series must terminate in order to achieve a finite solution. This only happens for special values of the separation constant λ_θ:

$$-\lambda_\theta = n[n-1] + 2[m+1]n + m + m^2 = [n+m][n+m+1] \tag{1-136}$$

When m is an integer it is customary to write the separation constant as $\lambda_\theta = -l[l+1]$ with $l \geq m$. When the separation constant is of this form, the functions $u(x)$ are polynomials, and the functions $b(x)$ are called the Associated Legendre Polynomials $P_l^m(x)$:

$$P_l^m(x) \equiv (-1)^m \left(1-x^2\right)^{m/2} \frac{\partial^m}{\partial x^m} P_l(x) = \frac{\left(1-x^2\right)^{m/2}}{2^l\, l!} \frac{\partial^{l+m}}{\partial x^{l+m}} \left(x^2-1\right)^l \tag{1-137}$$

with $0 \leq m \leq l$. Some authors omit the factor of $(-1)^m$ in this definition. Be careful!

The combined angular solutions $B(\theta)C(\phi)$ are called spherical harmonics. In normalized form and allowing for negative m the spherical harmonics are:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\left[\frac{[l-|m|]!}{l+|m|!}\right]} P_l^{|m|}(\cos\theta) e^{im\phi} \tag{1-138}$$

A few samples follow (from Jackson [1975] p.99). For $l=0$:

$$Y_{00}(\theta,\phi)=\sqrt{\frac{1}{4\pi}} \tag{1-139}$$

For l=1:

$$Y_{10}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{11}(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \tag{1-140}$$

For l=2:

$$Y_{20}(\theta,\phi)=\sqrt{\frac{5}{4\pi}}\left[\frac{3}{2}\cos^2\theta-\frac{1}{2}\right]$$

$$Y_{21}(\theta,\phi)=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_{22}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{8\pi}}\sin^2\theta\, e^{i2\phi} \tag{1-141}$$

For l=3:

$$Y_{30}(\theta,\phi)=\sqrt{\frac{7}{4\pi}}\left[\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right]$$

$$Y_{31}(\theta,\phi)=-\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta\left[5\cos^2\theta-1\right]e^{i\phi}$$

$$Y_{32}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{i2\phi}$$

$$Y_{33}(\theta,\phi)=-\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta\, e^{i3\phi} \tag{1-142}$$

In principle one could have half-integer values of l and m, but these are not generally believed to be useful since the integer-indexed Y_{lm}'s already form a complete set.

The wave equation may therefore be written in terms of eigenvalues:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]-\frac{l[l+1]}{r^2}+\frac{\omega^2}{c^2}\right\}A_{l,m,\omega}(r)=0 \tag{1-143}$$

This is called the Helmholtz equation.

1.5. Properties of Waves

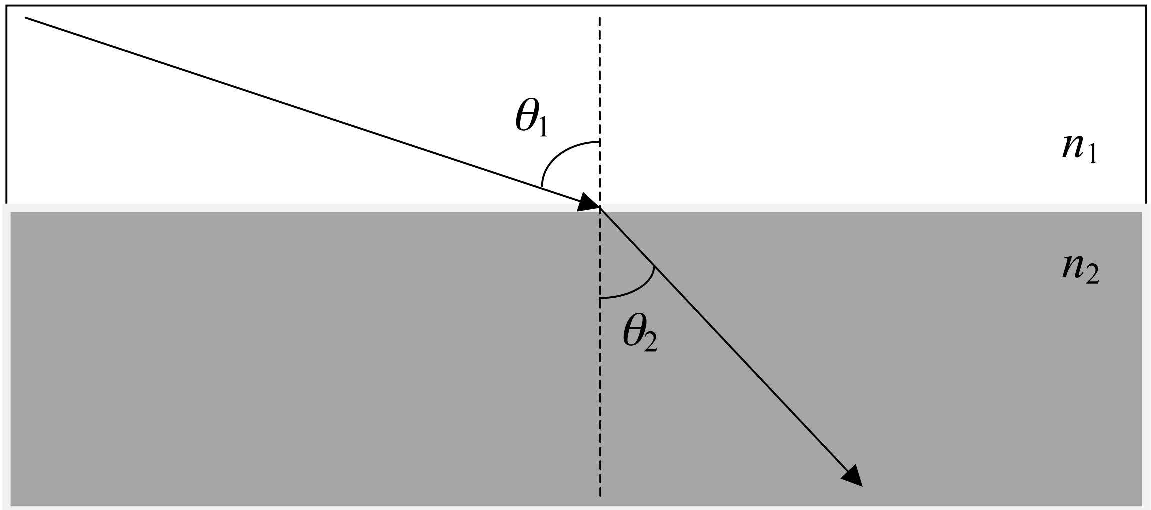

Figure 1.12 Diagram of a light ray propagating according to Snell's Law.

1.5.1. Wave propagation

Many readers are probably familiar with Snell's law for wave refraction at a boundary between two media:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \tag{1-144}$$

where n_i is the index of refraction which is defined as the ratio between the vacuum wave speed and the wave speed in the medium:

$$n_i = \frac{c_0}{c_i} \tag{1-145}$$

The angles are measured from perpendicular, so the wave direction is always closer to perpendicular in the region with slower speed (higher n). In other words, the wave bends toward the region of slower speed. It turns out that Snell's law, which applies only to distinct boundaries, is a special case of a more general formulation called Fermat's principle:

$$\delta \int \frac{dl}{c} = 0 \tag{1-146}$$

This condition means that the path of the wave between any two points is such that the integral of length divided by speed is an extremum (maximum, minimum, or inflection). Small changes in the path do not change the value of the integral to first order. For the case of waves propagating between points (x_1, y_1) and (x_2, y_2) and intersecting the boundary between regions 1 and 2 at the point (x,0), Fermat's principle becomes (see Figure 1.12 and Jenkins and White p. 14-18):

$$\delta\left[\frac{d_1}{c_1}+\frac{d_2}{c_2}\right]=\frac{\partial}{\partial x}\left[\frac{d_1}{c_1}+\frac{d_2}{c_2}\right]\delta x=\frac{\partial}{\partial x}\left[\frac{\left[\left[x-x_1\right]^2+y_1^2\right]^{1/2}}{c_1}+\frac{\left[\left[x_2-x\right]^2+y_2^2\right]^{1/2}}{c_2}\right]\delta x=0 \tag{1-147}$$

The derivative yields:

$$\left[\frac{1}{c_1}\frac{\left[x-x_1\right]}{\left[\left[x-x_1\right]^2+y_1^2\right]^{1/2}}+\frac{1}{c_2}\frac{\left[x_2-x\right]}{\left[\left[x_2-x\right]^2+y_2^2\right]^{1/2}}\right]\delta x=\left[\frac{\sin\theta_1}{c_1}+\frac{\sin\theta_2}{c_2}\right]\delta x=0 \tag{1-148}$$

This must be true when $\delta x \neq 0$, implying:

$$\frac{\sin\theta_1}{c_1}+\frac{\sin\theta_2}{c_2}=0 \tag{1-149}$$

Multiplication by the constant c_0 yields Snell's law.

1.5.2. Dispersion and Group Velocity

We have already seen that solutions of the wave equation have the form:

$$f(z,t)=f(z\pm ct) \tag{1-150}$$

The Fourier decomposition depends on (*z,t*) through the phase factor $\phi = kz \pm \omega t$. Setting the phase to a constant value yields $kz \pm \omega t = \phi_0$. Differentiation yields the phase velocity:

$$v_p=\frac{dz}{dt}=\mp\frac{\omega}{k} \tag{1-151}$$

If the phase velocity varies with frequency, the medium is said to be dispersive. Interference between different wave frequencies separates the wave into packets which have large oscillations where the phases of different frequencies are aligned and small oscillations where the phases of different frequencies are misaligned. The speed of these wave packets is called the group velocity, and can be derived from a simple example consisting of two frequency components (see e.g. Chen p. 69-70):

$$\begin{aligned}A_1(z,t)&=A_0e^{i\left[\left[k+dk\right]z-\left[\omega+d\omega\right]t\right]}\\A_2(z,t)&=A_0e^{i\left[\left[k-dk\right]z-\left[\omega-d\omega\right]t\right]}\end{aligned} \tag{1-152}$$

The sum of the two components is:

$$
\begin{aligned}
A_1(z,t) + A_2(z,t) &= A_0\left[e^{i\left[[k+dk]z-[\omega+d\omega]t\right]} + e^{i\left[[k-dk]z-[\omega-d\omega]t\right]}\right] \\
&= A_0 e^{i[kz-\omega t]}\left[e^{i[dkz-d\omega t]} + e^{-i[dkz-d\omega t]}\right] \\
&= A_0 e^{i[kz-\omega t]}\left[2\cos(dk\,z - d\omega t)\right]
\end{aligned}
\tag{1-153}
$$

The rapidly varying phase of the exponential factor propagates with the phase velocity (ω/k), while the slowly varying phase of the cosine factor propagates at the group velocity:

$$
v_g = \frac{d\omega}{dk} \tag{1-154}
$$

1.5.3. Interference and Diffraction

Waves from different sources, or waves from a single source but following different paths, will not generally have the same phase at the point where they combine. These phase differences result in the phenomenon of interference. Constructive interference occurs where the waves are in phase, so that the amplitudes are added. Destructive interference occurs where the waves are 180 degrees out of phase, so that the amplitudes are subtracted.
Diffraction is a form of interference which results from the wave propagating past a small object or through a small opening. Different path lengths for waves propagating past the object (or opening) from one side or the other produce the interference pattern.

1.5.4. Doppler shift

Frequency (ω') at detector moving away from fixed source (ω):

$$
\omega' = \omega(1 - v/c) \tag{1-155}
$$

Frequency (ω) at fixed detector from receding source (ω'):

$$
\omega = \frac{\omega'}{(1 + v/c)} \tag{1-156}
$$

1.5.5. Uncertainty Principle

Now consider a wave packet whose intensity has Gaussian shape with standard deviation σ_x at time t_0:

$$
\begin{aligned}
A(x,t_0) &= A_0 e^{-[x-x_0]^2/4\sigma_x^2} \\
A^2(x,t_0) &= A_0^2 e^{-[x-x_0]^2/2\sigma_x^2}
\end{aligned}
\tag{1-157}
$$

The Fourier representation is:

$$A(k,t_0) = \frac{1}{\sqrt{2\pi}} \int dx \; A_0 e^{-[x-x_0]^2/4\sigma_x^2} e^{-ik\cdot x}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \int dx \; A_0 e^{-[x-x_0]^2/4\sigma_x^2 - ik\cdot[x-x_0]}$$

$$= \frac{1}{\sqrt{2\pi}} A_0 e^{-\sigma_x^2 k^2} e^{-ikx_0} \int_{-\infty}^{\infty} dx \; e^{-([x-x_0]/2\sigma_x - ik\sigma_x)^2} \tag{1-158}$$

where the last step comes from completing the square in the exponent. This can be solved using a cute mathematical trick:

$$\int_{-\infty}^{\infty} dx \; e^{-x^2} = \left[\int_{-\infty}^{\infty} dx \; e^{-x^2} \int_{-\infty}^{\infty} dy \; e^{-y^2} \right]^{1/2} = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \; e^{-[x^2+y^2]} \right]^{1/2} = \left[2\pi \int_0^{\infty} r dr e^{-r^2} \right]^{1/2} = \sqrt{\frac{\pi}{2}} \tag{1-159}$$

Therefore:

$$A(k,t_0) = \sigma_x A_0 e^{-ikx_0} e^{-\sigma_x{}^2 k^2}$$

$$|A(k,t_0)|^2 = \sigma_x^2 A_0^2 e^{-2\sigma_x{}^2 k^2} \tag{1-160}$$

Note that the shape of the intensity profile (power spectrum) in the Fourier domain is a Gaussian with complex amplitude and standard deviation $\sigma_k = 1/2\sigma_x$. It turns out that the Gaussian shape minimizes the product of the standard deviations, so we can write the classical uncertainty relation as:

$$\sigma_x \sigma_k \geq 1/2 \tag{1-161}$$

where σ_x and σ_k now represent the standard deviations of arbitrary wave intensity profiles in the spatial and Fourier domains.

Thus far this uncertainty relation is simply a mathematical property of Fourier transforms. What makes it physically interesting is the fact that the waves also satisfy the wave equation, in which case the wave number k is inversely proportional to the wavelength. In three dimensions, the vector **k** indicates the propagation direction and is therefore related to the wave momentum. In particular, if the wave propagates through a slit of width Δx then the uncertainty in the x-component of the wave number is $\sigma_{k_x} \geq 1/[2\Delta x]$. Hence the uncertainty in wave propagation direction increases as the width of the slit decreases.

Similar uncertainty relations can be derived for other conjugate variables such as (ω,t). An interesting case is that of angular variables. Requiring the wave amplitude to be periodic in θ implies that the angular wave numbers m are quantized in integer steps:

$$f(\theta) = \sum_m F(m) e^{im\theta} \tag{1-162}$$

The Fourier transform is:

$$F(m) = \int f(\theta) e^{-im\theta} d\theta \tag{1-163}$$

The angular distribution cannot be a simple Gaussian because of the periodicity constraint. For illustration of the uncertainty principle for angular distribution, we will pick a distribution for which part of the Fourier transform (the cosine transform) can be performed analytically. Let:

$$f(\theta) = \frac{f_0 \left[1 - a^2\right]}{1 + a^2 - 2a\cos(\theta/2)} \tag{1-164}$$

As $a \to 1$ this distribution becomes sharply peaked at $\theta = \pi$.
The real part of the Fourier transform is:

$$f(m) = \int_0^{2\pi} f(\theta)\cos m\theta \, d\theta = f_0\left[1 - a^2\right]\int_0^{2\pi}\left[\frac{\cos m\theta}{1 + a^2 - 2a\cos(\theta/2)}\right]d\theta$$
$$= f_0\left[1 - a^2\right]\int_0^{\pi}\left[\frac{\cos 2m\theta'}{1 + a^2 - 2a\cos\theta'}\right]2d\theta' = 2f_0\pi a^{2m} \tag{1-165}$$

As $a \to 1$ the Fourier components drop off slowly. Hence the spread, or uncertainty, of m-values increases as the spread of angles decreases.

1.6. Summary

We have reviewed the basic concepts of classical mechanics, including the physical and mathematical properties of waves. Now we are ready to apply this knowledge to theories of matter.

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Figures

The following figures are believed to be free of copyright restriction, and were obtained from the sources listed. Other figures are either original works or are cited in the figure caption.

Figure 1.1 Aristotle (circa 384-322 BC).

Source: http://www.departments.bucknell.edu/history/carnegie/aristotle/bust.html

Figure 1.2 Aristotle's Universe. The Christian Aristotelian cosmos, engraving from Peter Apian's Cosmographia, 1524.

Figure 1.3 Ptolemy (AD 127-145).

Source: http://abyss.uoregon.edu/~js/glossary/ptolemy.html

Figure 1.5 Nicolaus Copernicus (1473-1643).

Source: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Copernicus.html

Figure 1.6 Johannes Kepler (1571-1630).

Source: http://www-history.mcs.st-and.ac.uk/Biographies/Kepler.html

Figure 1.7 Isaac Newton (1643 - 1727).

Source: http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Newton.html

Figure 1.8 Albert Einstein (1879 – 1955).

Source: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Einstein.html

Figure 1.10 Emmy Noether (1882-1935).

Source: www-gap.dcs.st-and.ac.uk/~history/Biographies/Noether_Emmy.html

Chapter 2. Matter Waves and Special Relativity

"Ignorance is preferable to error; and he is less remote from the truth who believes nothing, than he who believes what is wrong."
— Thomas Jefferson, *Notes on Virginia* (Query VI)

2.1. Introduction

"Science is the belief in the ignorance of experts."
— Richard Feynman [1969]

Early attempts at a wave theory of light presumed that light waves propagate through a universal medium in the same manner as sound waves through air. This medium was dubbed the luminiferous 'aether'. Christian Huygens [1690] [**Figure 2.1**] published an explanation of reflection and refraction based on the principle that each surface of a wave-front can be regarded as a source of secondary waves. Huygens also discovered that birefringent crystals can separate light rays into two distinct components (polarizations). Isaac Newton, among others, doubted the wave hypothesis in part because it could not explain this property of polarization. Nonetheless Newton did perceive a similarity between color and the vibrations which produce sound tones.

Figure 2.1 Christian Huygens (1629 – 1695)

In 1675 Olaf Roemer attributed variations in the observed orbital periods of Jupiter's moons to variable light propagation distance between Jupiter and Earth. This interpretation, combined with Giovanni Domenico Cassini's parallax determination of interplanetary distances in 1672, determined the speed of light to be about 2.1×10^8 m/s (recent measurements put the value at 2.99792×10^8 m/s).

Because light, unlike particles, propagates at a characteristic speed, Thomas Young [**Figure 2.2**] was convinced that light consists of waves. He demonstrated this wave nature by producing interference fringes from light passing through two narrow slits. Then in 1817 he explained polarization by proposing that light waves consist of transverse vibrations such as occur in an elastic solid. Augustin Fresnel [**Figure 2.3**] adopted Young's idea of transverse vibrations and developed a highly successful theory which explained diffraction and interference in addition to reflection and refraction. He supposed the aether to resist distortion in the same manner as a elastic solid whose density is proportional to the square of the refractive index.

Figure 2.2 Thomas Young (1773 – 1829)

Figure 2.3 Augustin Fresnel (1788 – 1827)

A conceptual problem with a solid aether is the question of how ordinary matter can coexist and move freely through it. George Gabriel Stokes [**Figure 2.4**] proposed that the aether was analogous to a highly viscous fluid or wax: elastic for rapid vibrations but fluid-like with respect to slow-moving matter.

Figure 2.4 George Gabriel Stokes (1819 – 1903)

Figure 2.5 James MacCullagh (1809 – 1847)

A more direct difficulty with the solid aether model was that density variations (e.g. at the interface between vacuum and medium) led to coupling between transverse and longitudinal waves, a phenomenon not observed for light waves. James MacCullagh [1839] [**Figure 2.5**] avoided this problem by proposing a 'rotationally elastic' aether whose potential energy Φ depends only on rotation (approximated by curl of displacement **a**):

$$\Phi = \frac{1}{2}\mu(\nabla \times \mathbf{a})^2$$

The resulting wave equation is:

$$\rho \frac{\partial^2 \mathbf{a}}{\partial t^2} = -\mu \nabla \times (\nabla \times \mathbf{a})$$

which is simply the equation of elastic shear waves which we derived in Chapter 1. Matter was now presumed to alter the elasticity of the aether rather than its density. This model successfully accounted for all of the known properties of light. Joseph Boussinesq [1868] [**Figure 2.6**] proposed that the aether could be regarded as an ordinary ideal elastic solid whose physical properties (density and elasticity) are unchanged by interaction with matter. The optical properties of matter were thus entirely due to the manner in which matter interacts with the aether. With this approach any classical optical phenomenon could be consistently modeled simply by finding the appropriate interaction term.

Figure 2.6 Joseph Boussinesq (1842-1929)

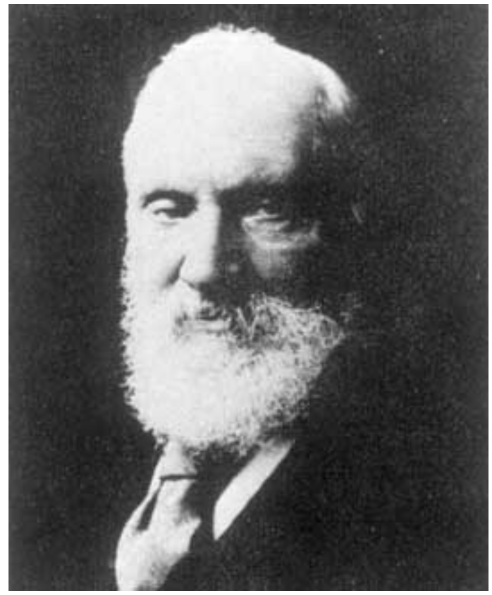


Figure 2.7 William Thomson (Lord Kelvin, 1824 – 1907)

In spite of these successes, scientists continued to pursue theories of a fluid aether through which solid matter could propagate. William Thomson (Lord Kelvin) [**Figure 2.7**] attempted to model the aether as a 'vortex sponge': a fluid full of small-scale vortices with initially random orientation. He argued that this system could support transverse waves analogous to those in an elastic solid. James Clerk Maxwell [1861a,b, 1862a,b] [**Figure 2.8**] modeled the aether as a network of rotating elastic cells interspersed with rolling spherical particles in order to derive the equations of electricity and magnetism. His resultant equations for light waves are equivalent to those of MacCullagh.

Figure 2.8 James Clerk Maxwell, 1831 – 1879

Figure 2.9 Albert Michelson, 1852-1931

Since matter was presumed to move through the aether as particles moving through a fluid, many attempts were made to directly measure the relative motion between the earth and the aether. The most notable of these was an experiment first reported by Albert Michelson [**Figure 2.9**] in 1881 and subsequently improved [Michelson and Morley 1887]. Interference fringes were formed by combining two beams of light which propagated along perpendicular paths. If the earth moves with respect to the aether then light propagating back and forth along a path aligned with the earth's motion should have a slightly slower average velocity than light propagating perpendicular to the earth's motion. Therefore the fringes should shift if the apparatus is rotated so that a given beam is alternately parallel and perpendicular to the direction of the earth's motion. However, no such effect was observed in this or other 'aether-drift' experiments.

Oliver Lodge [1893] demonstrated that the velocity of light is not noticeably affected by nearby moving matter, indicating that aether is not dragged along with matter. George FitzGerald proposed that the inability to measure motion relative to the aether could be explained if matter contracts along the direction of motion through the aether [Lodge 1892]. Joseph Larmor [1900] noted that in addition to the shortening of length, moving clocks should also run slower. Hendrik Lorentz [1904] [**Figure 2.10**] combined length contraction and time dilation to obtain the complete coordinate transformations. Henri Poincaré [1904] [**Figure 2.11**] gave the name 'Principle of Relativity' to the doctrine that absolute motion is undetectable. He also deduced that inertia increases with velocity and that no velocity can exceed the speed of light. Albert Einstein [1905a] reformulated relativity with the more positive assertion that the speed of light is a universal constant independent of observer motion.

Figure 2.10 Hendrik Lorentz (1853 – 1928)

Figure 2.11 Jules Henri Poincare (1854 – 1912)

One difficulty with the classical theory of light was a lack of success in describing radiation from a cavity at a fixed temperature (a 'black body'). Max Planck [1900] [**Figure 2.12**] derived the correct formula for blackbody radiation by supposing light to be emitted by vibrators whose energy $\varepsilon = nh\nu = n\hbar\omega$ is an integral multiple n of a constant h multiplied by the frequency ν (or a multiple of $\hbar = h/2\pi$ times the angular frequency $\omega = 2\pi\nu$). Albert Einstein [1905b] used the idea that radiation consists of discrete quanta in order to explain the photo-electric effect, in which the frequency of light must exceed a certain threshold in order to liberate electrons from a metal. Niels Bohr [1913] [**Figure 2.13**] used quantization of angular momentum and energy to derive energy levels and spectral frequencies of the hydrogen atom.

Figure 2.12 Max Planck (1858 – 1947)

Figure 2.13 Neils Bohr (1885-1962)

Recognizing that quantization is often associated with waves and vibrations, Louis Victor de Broglie [1924] [**Figure 2.14**] proposed in his doctoral thesis that electrons have a wave-like character with energy proportional to frequency $\varepsilon = \hbar\omega$ and momentum proportional to wave vector $\mathbf{p} = \hbar\mathbf{k}$. Bohr's quantization of angular momentum is then equivalent to the requirement that stable electron orbits contain an integral number of electron wavelengths. Walter Elsasser [1925] suggested that this wave property of electrons might explain maxima and minima in the angular distribution of electrons scattered from a platinum plate in experiments reported by Clinton Davisson and Charles Kunsman. The wave nature of electrons was confirmed in 1927 when electron diffraction by crystals was clearly demonstrated in experiments by Davisson and Lester Germer [1927] [**Figure 2.15**], and independently by George Thomson and A. Reid [1927]. See http://online.cctt.org/physicslab/content/phyapb/lessonnotes/dualnature/Davisson_Germer.asp for a comparison of diffraction using x-rays and electrons.

Figure 2.14 Louis Victor de Broglie, 1892-1987

Figure 2.15 Clinton Davisson and Lester Germer in 1927

The discovery of the wave-like propagation of matter actually solves the historic dilemma of how matter can move freely through a solid aether. In addition, the elastic medium itself need not change at all at the interface between vacuum and matter, thus explaining the lack of coupling to longitudinal waves. The wave nature of matter also leads directly to the Principle of Relativity without any modification of the classical Galilean view of Euclidean space and absolute time, as will be shown below. However, mechanical modeling of fundamental physical processes was no longer in vogue at the time of this discovery. Matter waves were not regarded as ordinary classical waves.

2.2. Measurements with waves

"If we are to achieve results never before accomplished, we must employ methods never before attempted."
— Francis Bacon

The first part of the following discussion closely follows Einstein's explanation of special relativity but with different rationale [Einstein 1956]. Let us consider the transformations between coordinates of relatively moving observers who measure distances by timing how long it takes for waves to propagate back and forth between two points. The defining equation would be:

$$(ds)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \sum_{i=1}^{3} (\Delta x_i)^2 = c^2 t_p^2 \qquad (2\text{-}1)$$

where *ds* is the spatial distance between two points at a fixed time, *c* is an arbitrary constant, and t_p is the time it would take to propagate a wave from one point to the other if they remained stationary. With this definition of distance, the constant *c* is simply a scaling factor which relates the units of distance to the units of time. This distance corresponds to the usual definition of distance if *c* is the speed of the wave used in the measurement.

Now suppose we consider propagation of a wave from point P_1 to point P_2. In a reference frame in which the points are stationary, Eq. 2-1 holds. An observer in a different inertial reference frame using the same definition of distance would have:

$$\sum_{i=1}^{3}\left(\Delta x_i'\right)^2 = c^2 t_p'^2 \tag{2-2}$$

The quantity $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2 t_p^2$ is therefore zero for both observers. Allowing for an arbitrary offset, the invariance of this quantity for different observers is precisely the condition which Lorentz used to derive the relativistic transformations. The quantity $\left(c^2 t^2 - x^2 - y^2 - z^2\right)^{1/2}$ is sometimes called the 'separation'.

For example, suppose a submarine navigator is using sonar both to measure time and to detect fish in the water. The sailors use special sonar clocks which measure time by cycling sound wave pulses back and forth across a fixed distance in the water perpendicular to the direction of motion. Each cycle of wave transmission, reflection, and detection at the original site of transmission constitutes a tick of the clock. In this analysis we will neglect any effects of displacement of water by moving submarines. An animated presentation of this analysis may be found at http://www.classicalmatter.org/UnderwaterRelativity.htm.

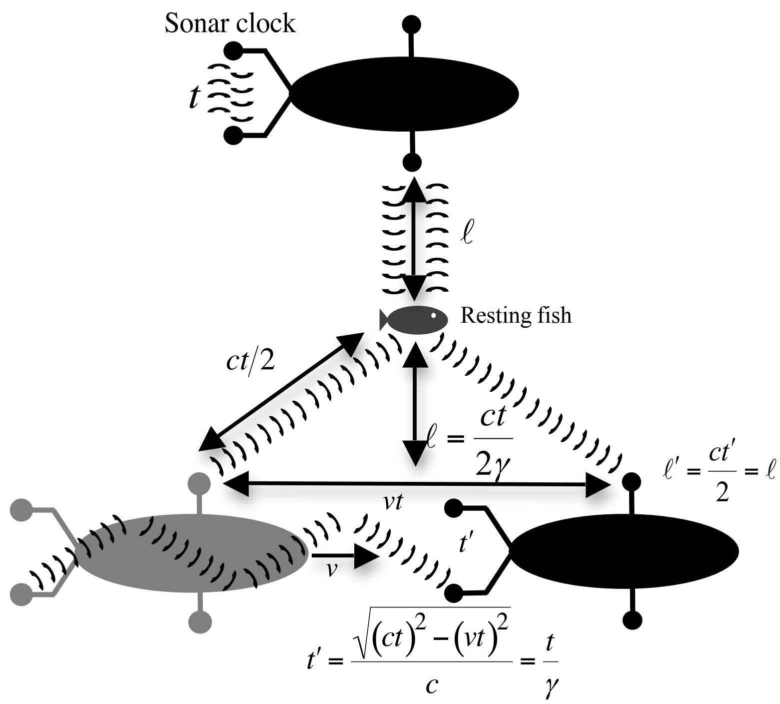


Figure 2.16 Time Dilation: The clock on O' ticks slower than the clock on O by the factor $\sqrt{1 - v^2/c_s^2}$ because waves travel farther between transmission and detection. Both O and O' measure the same number of clock cycles for a wave to propagate from their own sub to the fish and back. Hence they agree on distances perpendicular to the direction of relative motion.

2.2.1. Time dilation

If both the sub and the fish are at rest in the water, a sound wave reflected from the fish at distance ℓ would return after time $t = 2\ell/c_s$, where *cs* is the sound speed. The distance to the fish is therefore taken to be $\ell = c_s t/2$. Suppose now that the sub and fish are moving together in the water with common speed *v* perpendicular to the original direction of wave propagation (**Figure 2.16**). The path of the sonar clock waves forms two sides of a triangle for each cycle. A similar triangle is formed by the wave propagation to the fish and back. Therefore the number of clock ticks which occur during wave propagation to the fish and back is independent of speed. If the navigator doesn't realize that she is moving, she would assume the same relation between distance and time: $\ell' = \ell = c_s t'/2$. The navigator of a second submarine sitting still in the water would observe the wave propagate over a distance:

$$d \equiv c_s t = 2\sqrt{\ell^2 + \left(\frac{vt}{2}\right)^2} \tag{2-3}$$

Substituting $\ell \equiv c_s t'/2$ and solving for t' yields:

$$t' = t\sqrt{1 - v^2/c_s^2} \tag{2-4}$$

This equation merely expresses the fact that the clock on the moving submarine ticks more slowly that the stationary clock because the waves have farther to travel between ticks. Hence the time (*t*) measured by the stationary observer is longer than the time (t') measured by the moving observer. This phenomenon is referred to as 'time dilation'.

It is obvious that if the unprimed observer is truly stationary with respect to the water, then the moving clock does in fact tick more slowly. This is not merely an illusion. What is interesting is that the wave measurements performed by these submarines are insufficient to determine which sub is actually moving with respect to the water. Therefore the moving sub would interpret the stationary clock as running slowly, and in this case the effect is an illusion. This point will be discussed below in connection with Doppler shifts.

Since the stationary navigator sees the fish (and first sub) move a distance $x=vt$ while the wave is propagating, the above equation can be rewritten as:

$$t' = \frac{t\left(1 - v^2/c_s^2\right)}{\sqrt{1 - v^2/c_s^2}} = \frac{t - vx/c_s^2}{\sqrt{1 - v^2/c_s^2}} \tag{2-5}$$

which is the Lorentz transformation of time between two observers, with the primed observer moving in the *x*-direction with velocity +*v* with respect to the unprimed observer.

2.2.2. Length contraction

Since both observers measure the same distance $\ell' = \ell$, the transformation of coordinates perpendicular to the motion must be simply:

$$y' = y$$

$$z' = z$$

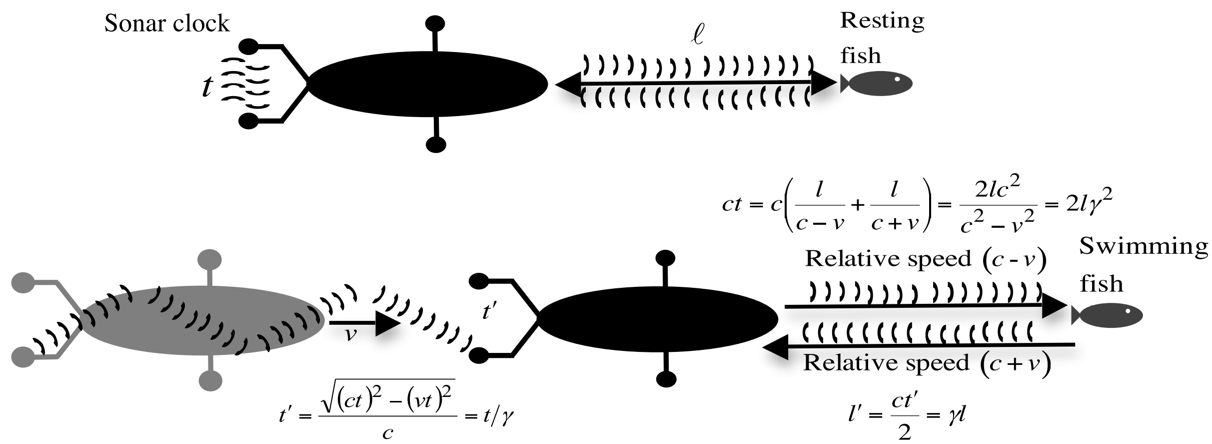


Figure 2.17: Length Contraction: The true wave propagation time for the co-moving sub and fish is longer than for the stationary sub and fish by the factor $1/\left(1 - v^2/c_s^2\right)$. Since the moving clock runs slow, the perceived propagation time is longer only by the factor $1/\sqrt{1 - v^2/c_s^2}$. Hence the stationary sub observes a shorter length than the moving sub.

Now suppose that the first sub and fish are moving relative to the second sub parallel to the direction of wave propagation [**Figure 2.17**].

As seen by the stationary sub, the frequency of the sonar clock on the first sub is slow according to Eq. 2-4 since the measured time t' is proportional to the moving clock frequency ω' times the absolute time *t*:

$$\omega' = \omega\sqrt{1 - v^2/c_s^2} \tag{2-6}$$

The absolute distance between the fish and sub remains constant at ℓ. However the relative speed between the outgoing wave and the target fish is (*c*-*v*) whereas the relative speed between the sub and the incoming wave is (*c*+*v*). Therefore the propagation time is:

$$t = \frac{\ell}{\left(c_s + v\right)} + \frac{\ell}{\left(c_s - v\right)} = \frac{2\ell}{c_s\left(1 - v^2/c_s^2\right)} \tag{2-7}$$

Of course the moving sub still uses the relation $\ell' = c_s t'/2$. Substituting the temporal relation $t'/t = \sqrt{1 - v^2/c_s^2}$ yields the relation between lengths:

$$\ell' = \frac{c_s t\sqrt{1 - v^2/c_s^2}}{2} = \frac{\ell}{\sqrt{1 - v^2/c_s^2}} \tag{2-8}$$

The stationary observer measures a shorter length than the moving observer. This phenomenon is known as length contraction. In this case the moving observer measurement is artificially long due to the fact that the actual sound velocity relative to the observer is not the same for the outgoing and incoming directions. Since the wave propagates for a longer time in the direction of slower relative motion, the effect is an apparent increase in length relative to a stationary observer. Again, however, it is important to realize that the wave measurements alone do not determine which observer is moving.

As noted previously, the origin of the moving frame corresponds to $x=vt$ in the stationary frame. Therefore the coordinate transformation is obtained by $\ell' \to x'$ and $\ell \to x - vt$:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c_s^2}} \qquad (2\text{-}9)$$

which is the Lorentz transformation of position along the direction of motion.

It is customary to use the definitions:

$$\begin{aligned} \beta &= v/c_s \\ \gamma &= \left(1 - v^2/c_s^2\right)^{-1/2} \end{aligned} \qquad (2\text{-}10)$$

A useful identity is:

$$\gamma^2 = \left(1 - \beta^2\right)^{-1} = 1 + \beta^2\gamma^2 \qquad (2\text{-}11)$$

Using the above expressions, the Lorentz transformations become:

$$\begin{aligned} c_s t' &= \gamma c_s t - \beta\gamma x \\ x' &= \gamma x - \beta\gamma c_s t \\ y' &= y \\ z' &= z \end{aligned} \qquad (2\text{-}12)$$

where subscripts are used to emphasize that we are discussing sound waves.

The inverse transformations merely change the sign of v (or β):

$$\begin{aligned} c_s t &= \gamma c_s t' + \beta\gamma x' \\ x &= \gamma x' + \beta\gamma c_s t' \\ y &= y' \\ z &= z' \end{aligned} \qquad (2\text{-}13)$$

Thus we see how Lorentz transformations can be obtained by using sonar or any other type of wave to measure time and distance. Lorentz invariance is not a property of time and space *per se*. Rather it results from the methods used to measure time and distance. If the above-mentioned sailors were to rendezvous to share their data and some vodka, they might conclude after a few drinks that absolute time and space in moving underwater reference frames are related by Lorentz transformations using the speed of sound in water. After sobering up, however, they would realize that sonar is not the only way to measure time and distance and that their measurements are not evidence of any non-classical properties of underwater space-time.

2.2.3. Length and time standards

The sonar clock might seem like an odd sort of clock, but consider the standard definition of a second, which is 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom [Taylor 1995]. If we regard the cesium atom as a kind of optical cavity which resonates at the prescribed frequency, then this is quite similar to our sonar clock.

Consider also that the standard definition of the meter is the length of the path traveled by light in vacuum during a time interval of $1/c$ =1/299,792,458 of a second [Taylor 1995]. So we do in fact equate length with wave propagation time just as our hypothetical sailors do, and the quantity c is nothing more than a unit conversion factor.

2.2.4. Doppler shift

Thus far we have shown that when waves are used to measure distance and time, the space-time coordinates transform between relatively moving observers according to the Lorentz transformations. Transformation of other dynamical variables is straightforward.

The phase of a plane wave is given by:

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t \tag{2-14}$$

This quantity is independent of observer motion. Therefore:

$$\mathbf{k}' \cdot \mathbf{x}' - \omega' t' = \mathbf{k} \cdot \mathbf{x} - \omega t$$

For motion along the x-axis we can plug in the inverse transformations for x and t to obtain:

$$\begin{aligned} k'_x x' - \omega' t' &= k_x \left(\gamma x' + \beta\gamma c_s t'\right) - \omega\left(\gamma t' + \beta\gamma\, x'/c_s\right) \\ k'_y y' &= k_y y \\ k'_z z' &= k_z z \end{aligned} \tag{2-15}$$

The coefficients of t' must be equal on both sides of the equation, and likewise for the coefficients of x'. Therefore:

$$\begin{aligned} \omega' &= \gamma\omega - \beta\gamma\, c_s k_x \\ k'_x &= \gamma k_x - \beta\gamma\, \omega/c_s \\ k'_y &= k_y \\ k'_z &= k_z \end{aligned} \tag{2-16}$$

Letting $\boldsymbol{\beta} = \mathbf{v}/c$, the transformation for arbitrary direction of relative velocity is:

$$\begin{aligned} \omega' &= \gamma\left(\omega - \boldsymbol{\beta} \cdot c\mathbf{k}\right) \\ c_s k'_{\|} &= \gamma\left(c_s k_{\|} - \beta\omega\right) \\ \mathbf{k}'_{\perp} &= \mathbf{k}_{\perp} \end{aligned} \tag{2-17}$$

Hence the spatio-temporal frequency components ($\omega, c\mathbf{k}$) transform in the same manner as the coordinates (ct,$\mathbf{x}$). Quantities which transform according to these Lorentz transformations are called ‘four-vectors’. Each four-vector has three spatial components and a temporal component. Other examples of four-vectors (with respect to light waves) include:

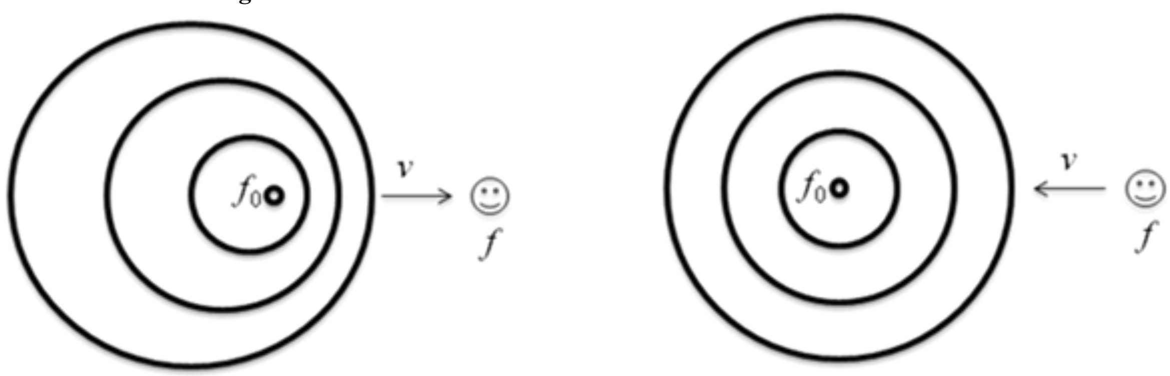
$(\gamma c, \gamma \mathbf{v})$	Four – velocity
$(E/c = \gamma m_0 c, \mathbf{p} = \gamma m_0 \mathbf{v})$	Energy, momentum
$(\rho c, \mathbf{J})$	Electromagnetic charge, current

Note that for light waves $|c\mathbf{k}| = \omega$. Hence the frequency and wave vector transformations for motion parallel to **k** can be written as:

$$\omega' = \gamma\omega(1-\beta) = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$
$$k'_{||} = \gamma k_{||}(1-\beta)$$
$$\mathbf{k}'_{\perp} = \mathbf{k}_{\perp} = 0 \qquad (2\text{-}18)$$

The first of these equations is the relativistic Doppler shift formula for light waves.

The relativistic Doppler shift has a simple interpretation. First, consider the classical Doppler shifts as shown in **Figure 2.18** below.



$$f = \frac{c}{\lambda} = \frac{c}{\left(\frac{c-v}{f_0}\right)} = f_0\left(\frac{1}{1-\beta}\right) \qquad f = \frac{c+v}{\lambda} = \frac{c+v}{\left(\frac{c}{f_0}\right)} = f_0(1+\beta)$$

Figure 2.18 Classical Doppler shifts for moving (approaching) source and detector differ by a factor of $[1+\beta][1-\beta] = 1/\gamma^2$. This factor is not affected by reversal of the velocity direction.

Consider a stationary observer O in a lighthouse which pulsates with angular frequency ω. An observer O' moves away from the lighthouse starting at t=0 in a speedboat. As a moving detector, O' receives a classically Doppler-shifted frequency of $\omega(1-\beta)$. However, O''s clock is running slow by the factor $1/\gamma$ because the boat is moving. Hence O' perceives the incident wave frequency to be higher by the factor γ so that $\omega' = \gamma\omega(1-\beta)$. The stationary observer O would agree with this correct description of events. Note that observer O can measure the speed of observer O' by measuring the time of flight of radar pulses which reflect off of O' and back to O. Successive pulses separated by transmission time interval τ_T will be received with delay time interval $\tau_R = \tau_T(1 + v/c)$, yielding $v = c(\tau_R - \tau_T)/\tau_T$.

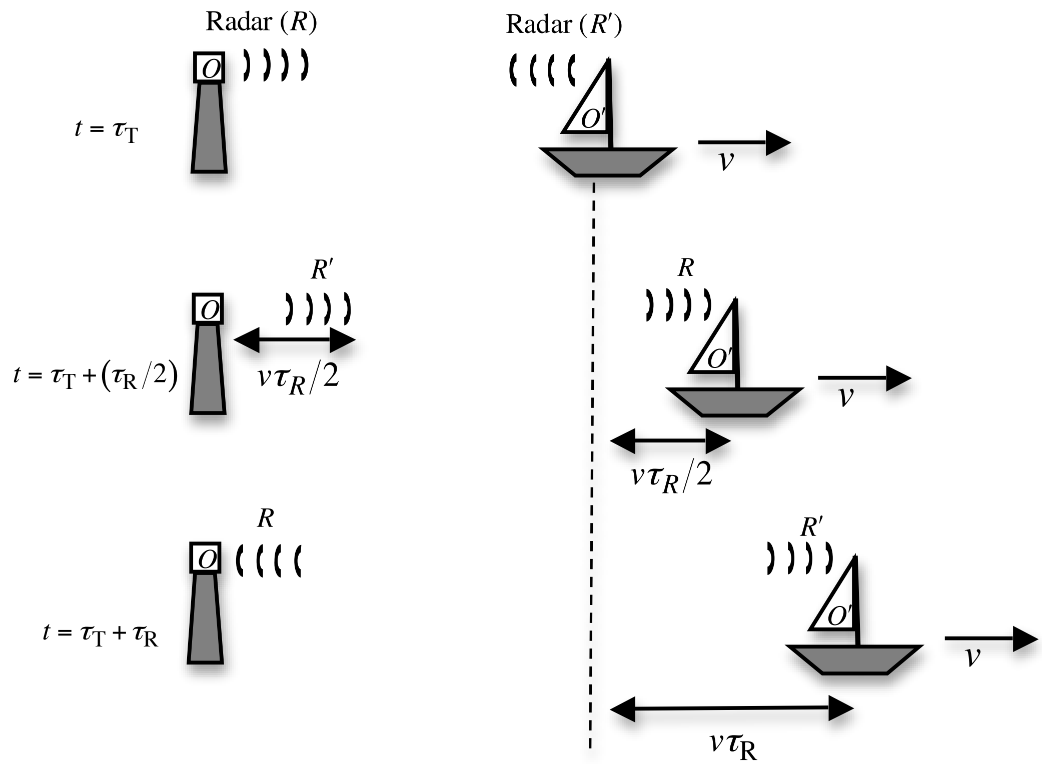


Figure 2.19 Velocity Measurement: Radar signals sent simultaneously by *O* and O' will also be received simultaneously after reflection. Although O' 's clock ticks slowly, the proportionality between radar pulse propagation time and total time elapsed is the same as for *O*. Therefore both *O* and O' measure the same relative velocity.

Conversely, the observer O' incorrectly believes that he is stationary and that *O* is moving. O' measures the speed of recession of the lighthouse via radar. The true propagation time of the each pulse is the same as measured by *O* (see **Figure 2.19** above). The fact that O' 's clock is running slowly reduces all of his measured times by the factor $1/\gamma$, but this does not affect the proportionality between the transmission time interval and the reception time interval. Therefore O' sees *O* recede with speed *v*.

Observer O' observes the lighthouse light fluctuate with frequency $\omega' = \gamma(1-\beta)$. This formula accounts for slowing of the moving clock and Doppler shift at the moving (receding) receiver. O' presumes the detected frequency to be classically Doppler shifted at the source by a factor of $1/(1+\beta)$. Correcting for this Doppler shift yields $\omega'(1+\beta)$ for the co-moving source frequency. Since O' thinks that *O*'s clock is slow, the correction factor γ is again introduced to obtain the frequency perceived at the source. This leads to:

$$\omega = \gamma\omega'(1+\beta) = \omega'\left(\sqrt{\frac{1+\beta}{1-\beta}}\right) \tag{2-19}$$

which is of course the inverse frequency transformation. Note that O' incorrectly attributes the Doppler shift to a moving source rather than a moving detector, resulting in an erroneous factor of $(1+\beta)(1-\beta)=1/\gamma^2$. However, this mistake is exactly compensated by the fact that O' incorrectly believes that O's clock is running slower by the factor $1/\gamma$ when in fact it is running faster by the factor γ . O' mistakenly multiplies by γ when he should have divided by gamma to correct for the different clock rates (an erroneous factor of γ^2). The erroneous factors of γ^2 and $1/\gamma^2$ cancel and O' correctly deduces the frequency ω for the stationary source at O. This cancellation of errors renders impossible the determination of motion relative to the medium which carries the wave. It is the crux of special relativity.

If the relative motion is not along the line of separation then the Doppler shifts are dependent on angle. Nonetheless, one can correct for this angular dependence to determine the head-on Doppler shift consistent with the analysis above.

2.3. Matter waves and light

"It is better to light one small candle than to curse the darkness."
(*"與其詛咒黑暗, 不如然起蠟燭"*)
— Confucius (孔夫子)

One limitation of the above discussion is that sound waves in water are too simple to serve as a model of matter. The sonar clock had to be oriented perpendicular to the direction of motion so that its apparent length was independent of velocity. Another problem is that sound waves are scalar waves, described by a single number (e.g. pressure) at each point. A more interesting medium to consider is an elastic solid, which can support shear waves whose amplitude (displacement or rotation) can have multiple components. Waves which include significant rotations are especially of interest because this allows for intrinsic, or spin, angular momentum in addition to the orbital angular momentum associated with propagation of the wave.

The above results show that the equations of special relativity are applicable to a wide variety of wave phenomena. The Lorentz transformations relate wave measurements made in different frames of reference. It is well-known (and easily verified) that any wave equation of the form:

$$\left[\frac{\partial^2}{\partial t^2}-c^2\nabla^2+M^2\right]f=0 \tag{2-20}$$

with invariant scalar M is invariant under Lorentz transformations with wave speed c. In other words Lorentz invariance is a general property of waves and not specific to electromagnetic waves.

Now we are in a position to appreciate what is special about light. Ordinarily we do not measure distances and times by propagating waves back and forth. Instead we use material clocks and rulers. The amazing thing about material clocks and rulers is that the resulting distance and time measurements transform with exactly the same Lorentz transformations as would be obtained if the measurements had been made by propagating light waves. In other words, matter behaves as if it consists of waves which propagate at the speed of light. Since matter can appear to be stationary, we must suppose that the waves somehow propagate in cyclic paths in the 'rest' frame. Such waves are commonly referred to as soliton waves.

Historically, the equations of relativity were derived from the observation that absolute motion is undeterminable. Einstein reformulated relativity on the basis that the speed of light is independent of observer motion. Yet now we have a simpler alternative postulate for special relativity: *matter consists of waves which propagate at the speed of light*. This physical picture suggests that matter and anti-matter can annihilate into photons and *vice versa* because photons and matter are simply different packets of the same type of wave. We will see that our new hypothesis is also consistent with the Dirac equation for the electron, in which the velocity operator has eigenvalues of magnitude *c*. Mass is associated with a reduction in group velocity which may be attributed to rotation of the wave propagation direction.

With respect to aether-drift experiments such as performed by Michelson and Morley, it is clear that if matter waves have the same speed as light waves then any effect of earth's propagation through the vacuum would equally affect the light waves and the apparatus used to measure them. It has long been recognized that Lorentz invariance of matter is required to explain the null result of such experiments. What has not been generally recognized (though there are numerous exceptions) is that the wave nature of matter provides the basis for relativity and is entirely consistent with classical notions of absolute space and time.

2.3.1. Soliton waves

Let c represent the characteristic speed of transverse waves in an elastic medium. The equation of evolution of the wave amplitude $\mathbf{a}(\mathbf{x},t)$ is:

$$\partial_t^2 \mathbf{a} = c^2 \nabla^2 \mathbf{a} - \mathbf{u} \cdot \nabla \dot{\mathbf{a}} + \mathbf{w} \times \dot{\mathbf{a}} \tag{2-21}$$

Assume that the convection and rotation terms reduce to a constant coefficient of $\mathbf{a}$, so that each component satisfies:

$$\frac{\partial^2 a_i}{\partial t^2} = \left(c^2 \nabla^2 - M^2\right) a_i$$

It is common to use Fourier decomposition so that the wave equation can be written as:

$$\omega^2 A_i = \left(c^2 k^2 + M^2\right) A_i \tag{2-22}$$

where $Ai(\mathbf{k},\omega)$ is the Fourier transform of the wave amplitude $ai\ (\mathbf{x},t)$. The wave group velocity u is given by:

$$u = \frac{d\omega}{dk} = \frac{k}{\omega} c^2 = \frac{\left(\omega^2 - M^2\right)^{1/2}}{\omega} c \tag{2-23}$$

Solving for $k = \omega(k) u / c^2$ yields:

$$k = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \frac{M}{c^2} u = \gamma \frac{M}{c^2} u \tag{2-24}$$

where we have used the familiar definition of γ to obtain the expression on the right.
Substitution into the wave equation yields:

$$\omega = M\left[1+\gamma^2\frac{u^2}{c^2}\right]^{1/2} = \gamma M \tag{2-25}$$

If we define $M \equiv m_0c^2/\hbar$ then we obtain the quantum mechanical relations:

$$\hbar k = \gamma m_0 u$$
$$\hbar\omega = \gamma m_0 c^2 \tag{2-26}$$
$$[\hbar\omega]^2 = \frac{[m_0c^2]^2}{1-u^2/c^2} = \frac{[m_0uc]^2+[m_0c^2]^2-[m_0uc]^2}{1-u^2/c^2} = [\hbar kc]^2+[m_0c^2]^2$$

2.3.2. Energy and momentum

A special property of electron waves, which will be discussed in Chapter 3, is that the energy is proportional to frequency ($E = \hbar\omega$) and momentum is proportional to the wave vector ($\mathbf{p} = \hbar\mathbf{k}$). Classically, the quantity $\hbar$ must represent the integrated wave amplitude. We assume that all matter waves have similar proportionalities, though perhaps with different integrated wave amplitudes. Using these substitutions yields in the above equations yields the relativistic relations:

$$\mathbf{p} = \gamma m_0\mathbf{u}$$
$$E = \gamma m_0c^2 \tag{2-27}$$
$$E^2 = p^2c^2 + m_0^2c^4$$

This last equation, given the first two, merely expresses the tautology:

$$c^2 = u^2 + \left(c^2 - u^2\right) = u^2 + \frac{c^2}{\gamma^2} \tag{2-28}$$

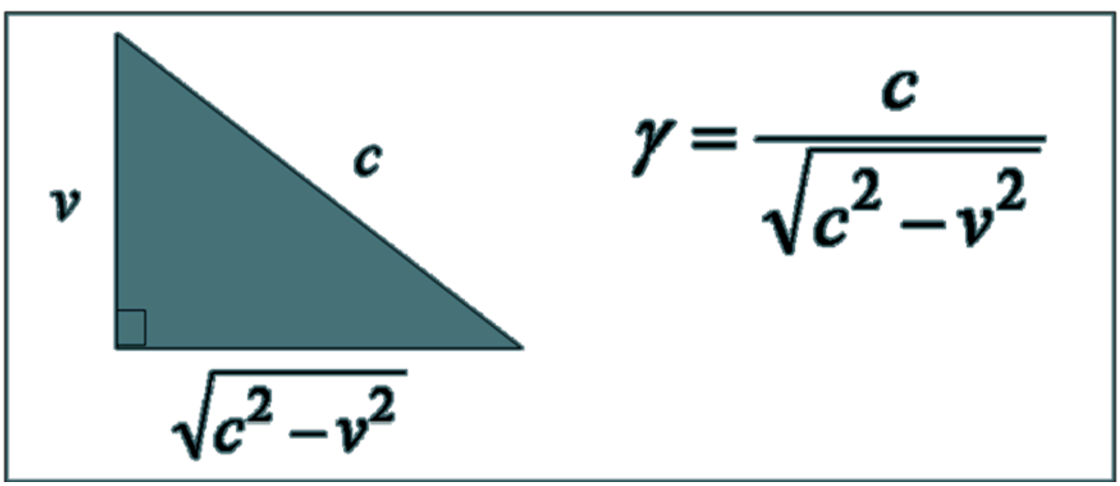


Figure 2.20 This is the Pythagorean relation for a right triangle with sides (c, v, $\sqrt{c^2-v^2}$).

The hypotenuse c, which corresponds to energy, indicates that the disturbance moves with speed c. The velocity v corresponds to momentum and indicates propagation in the direction of the wave vector. The velocity $\sqrt{c^2 - v^2}$ corresponds to mass and indicates propagation perpendicular to the wave vector (or at least independently from the wave vector: the Pythagorean relation also holds, on average, for cycloidal motion, e.g. $\mathbf{u} = \hat{\mathbf{x}}_\perp u_\perp \cos\theta + \hat{\mathbf{x}}_{||}(u_\perp \sin\theta + u_{||})$ with $u_{||} = v$ and $|\mathbf{u}| = c$). Since the propagation associated with mass does not yield any net transport of the disturbance, it must be at least approximately periodic, and the simplest assumption is circular motion. The general propagation of the wave would then be helical or cycloidal (or in between). Hestenes [1990] has also proposed helical motion of elementary particles.

Multiplying each side of the above velocity triangle by $\gamma m_0 c$ yields the energy-momentum relations.

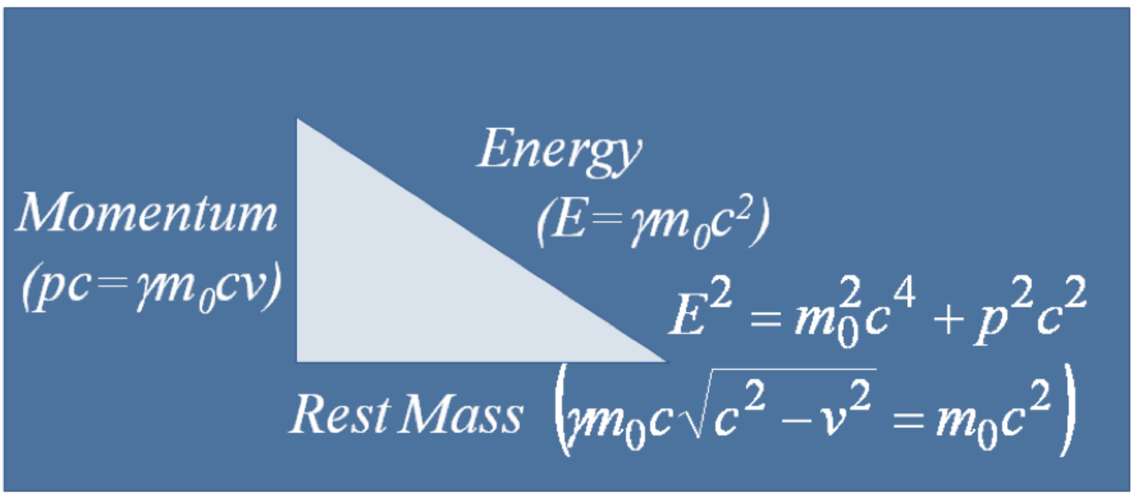


Figure 2.21 Triangular relationship between rest mass, momentum, and energy.

If the stationary frequency of an elementary particle is really associated with circular motion then we can compute the radius of the motion. For electrons we have:

$$R_c = \frac{c}{\omega} = \frac{\hbar c}{m_0 c^2} = 3.8616 \times 10^{-11} \text{ cm} \tag{2-29}$$

Note that this quantity is different from the Bohr radius ($R_e = \hbar^2 / m_0 e^2 = 5.2918 \times 10^{-9}$ cm) which is the classical radius of the electron orbit in the ground state of the hydrogen atom. The ratio between these two distances is called the fine structure constant:

$$\alpha = \frac{R_c}{R_e} = \frac{e^2}{\hbar c} = \frac{1}{137.04}$$

The definitions of E and p_i lead directly to the equation of motion $k_i E = \omega p_i$ in the Fourier domain. In the spatial domain this is the classical relationship between kinetic energy and momentum:

$$\frac{\partial E}{\partial x_i} = \frac{\partial p_i}{\partial t} \tag{2-30}$$

2.3.3. Transformation of velocity

The expression for group velocity can be combined with the transformation laws for frequency and wave vector to work out the transformation properties of the velocity. For relative motion parallel to the velocity only the component $k_{||}$ is affected:

$$u'_{||} = c^2 \frac{k'_{||}}{\omega'} = c^2 \frac{\gamma k_{||} - \beta\gamma\, \omega/c}{\gamma\omega - \beta\gamma c k_{||}} = c\frac{1-\beta\, c/u}{c/u - \beta} = \frac{u-\beta c}{1-\beta\, u/c} = \frac{u-v}{1-uv/c^2} \tag{2-31}$$

This is the transformation law for velocity parallel to the direction of relative motion. For relative motion perpendicular to the velocity the only change is to ω:

$$u'_{\perp} = c^2 \frac{k'_{\perp}}{\omega'} = c^2 \frac{k_{\perp}}{\gamma\omega} = \frac{u}{\gamma} \tag{2-32}$$

For an arbitrary direction of relative motion **v**, we use $u_{||} = \mathbf{u}\cdot\mathbf{v}/v$ to obtain the transformation laws for components of velocity parallel ($u_{||}$) and perpendicular ($\mathbf{u}_{\perp}$) to the direction of relative motion:

$$u'_{||} = c^2 \frac{\gamma k_{||} - \beta\gamma\, \omega/c}{\gamma\omega - \boldsymbol{\beta}\cdot\mathbf{k}\gamma c} = c^2 \frac{u_{||} - \beta c}{c^2 - c\boldsymbol{\beta}\cdot\mathbf{u}} = \frac{u_{||} - v}{1 - \mathbf{v}\cdot\mathbf{u}/c^2}$$

$$\mathbf{u}'_{\perp} = c^2 \frac{k_{\perp}}{\gamma\omega - \boldsymbol{\beta}\cdot\mathbf{k}\gamma c} = c^2 \frac{k_{\perp}}{\gamma\left[\omega - \boldsymbol{\beta}\cdot\mathbf{u}\omega c\right]} = \frac{\mathbf{u}_{\perp}}{\gamma\left[1 - \mathbf{u}\cdot\mathbf{v}/c^2\right]}$$

$$u'_{||} = \frac{u_{||} - v}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}$$

$$\mathbf{u}'_{\perp} = \frac{\mathbf{u}_{\perp}}{\gamma\left(1 - \mathbf{u}\cdot\mathbf{v}/c^2\right)} \tag{2-33}$$

2.3.4. The twin paradox

One supposedly non-intuitive consequence of relativity is that two twins can change their relative age through motion. If one twin (Theo=*O*) remains stationary while the other twin

(Primo= O') takes a high-speed journey through space, then the twin who traveled will return younger that the twin who stayed home. A more common manifestation of this phenomenon is that high-energy cosmic ray particles which zoom to earth at relativistic speeds have longer lifetimes than otherwise identical slow-moving particles. Although the effect of motion on time may seem almost magical, the explanation is really quite simple.

Consider a clock which counts the number of circular orbits executed by an electron wave. Any clock made of matter waves will tick at a proportionate rate. While the stationary electron executes a circular path, a moving electron executes a spiral (or cycloidal) path with the same absolute speed *c*. Since the moving electron travels farther than the stationary electron during each rotation cycle, a moving electron clock ($t' = \omega'\tau$) will tick more slowly than a stationary one ($t = \omega\tau$). For a translational velocity of *v*, the speed of circulation is:

$$v'_\perp = \left(c^2 - v_\parallel^2\right)^{1/2} = c/\gamma \tag{2-34}$$

and therefore the moving clock ticks more slowly ($t' < t$) by the factor:

$$\frac{t'}{t} = \frac{v'_\perp}{v_\perp} = \frac{v'_\perp}{c} = 1/\gamma \tag{2-35}$$

This is equivalent mathematically and similar physically to the derivation above of time dilation for sound waves in water. Hence the moving Primo will age less than the stationary Theo.

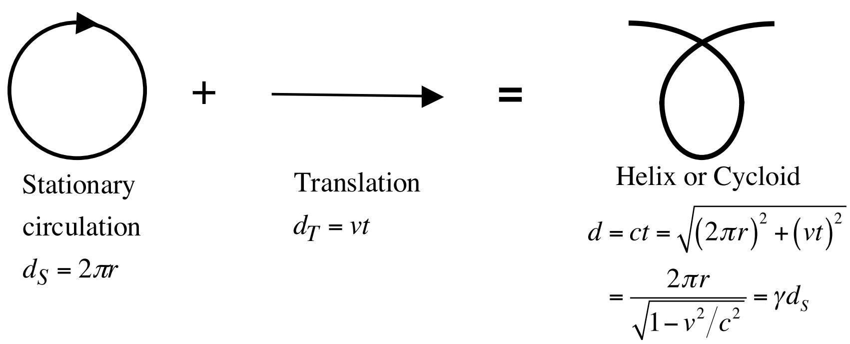


Figure 2.22

Time Dilation: Moving matter waves propagate farther than stationary matter waves during each cycle. Therefore moving clocks tick more slowly than stationary clocks. d_S = distance traveled in one cycle of stationary wave, d_T = translational distance. The distance formula for the cycloid is exact only for an integer number of cycles.

We have stated before that wave measurements cannot determine absolute motion relative to the medium. Therefore Primo should end up younger than Theo even if they are initially moving with respect to the medium. Suppose that the two twins Primo and Theo are initially moving together with velocity v_1 in the *x* direction. A stationary observer sees Primo slow to a stop at *t*=0, wait for a time $t = T_1$, then accelerate to speed v_2 to catch up with Theo at time $t = T_1 + T_2 = T$. In this case Primo is actually aging more rapidly than Theo at first, but then ages very slowly while trying to catch up. Note that:

$T_1 = T(1 - v_1/v_2)$
$T_2 = T(v_1/v_2)$
At the time the twins meet up again, Theo has aged by T/γ_1 since his clock is running slower than a stationary clock (using $\gamma_i = (1 - v_i^2/c^2)^{-1/2}$). But Primo has aged by $T_1 + T_2/\gamma_2 = T(1 - v_1/v_2 + v_1/\gamma_2 v_2)$. The difference in their ages is therefore:

$$T_{Theo} - T_{Primo} = T\left(\frac{1}{\gamma_1} - 1 + \frac{v_1}{v_2}\left(1 - \frac{1}{\gamma_2}\right)\right) \tag{2-36}$$

To second order in *v/c* terms, this difference is:

$$T_{Theo} - T_{Primo} \approx T\left(1 - \frac{1}{2}\frac{v_1^2}{c^2} - 1 + \frac{v_1}{v_2}\left(1 - \left(1 - \frac{1}{2}\frac{v_2^2}{c^2}\right)\right)\right) = T\left(-\frac{1}{2}\frac{v_1^2}{c^2} + \frac{1}{2}\frac{v_1 v_2}{c^2}\right) \geq 0 \tag{2-37}$$

where the inequality arises from the fact that $v_2 \geq v_1$. More generally, we can try to minimize the age difference with respect to v_2 (for a given *T* and v_1). The minimization condition is:

$$0 = \frac{d}{dv_2}\left(\frac{1}{\gamma_1} - 1 + \frac{v_1}{v_2}\left(1 - \frac{1}{\gamma_2}\right)\right) \tag{2-38}$$

which yields after a little algebra:

$$\frac{v_1}{v_2}\left(1 - \frac{1}{\gamma_2}\right) = \frac{v_1 v_2}{c^2}\gamma_2 \tag{2-39}$$

Substitution of this expression into the time difference yields:

$$T_{Theo} - T_{Primo} \geq T\left(\frac{1}{\gamma_1} - 1 + \frac{v_1 v_2}{c^2}\gamma_2\right) \tag{2-40}$$

Since $\gamma_2 v_2 \geq v_1$ the inequality can be written as:

$$T_{Theo} - T_{Primo} \geq T\left(\frac{1}{\gamma_1} - 1 + \frac{v_1^2}{c^2}\right) \geq T\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_1^2}\right) \geq 0 \tag{2-41}$$

since $\gamma_1 \geq 1$. Hence the twin who moves away and comes back always ages less than the twin whose motion was constant. This is a simple consequence of the wave nature of matter.

2.4. Alternative interpretations

> *"A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding."*
> — Isaac Newton

The reader should be warned that the simple interpretation of relativity presented here is not generally understood. Since its inception at the dawn of the 20th century, the Principle of Relativity has been interpreted as a physical law rather than as a purely mathematical relationship between space and time measurements. It is believed that geometrical relationships between measurements accurately represent the geometry of physical space. Such an interpretation assumes that measurements of distance and time can approach perfection. The four-dimensional space-time that satisfies the principle of relativity is usually referred to as "Minkowski space". According to our point of view, Minkowski space is the space of measurements made with waves propagating in a Galilean physical space-time.

It has long been recognized that compliance with the Principle of Relativity requires matter waves to be Lorentz covariant. However the converse logic has been largely ignored. Lorentz covariance is a property of waves, and the wave nature of matter implies the Principle of Relativity for a classical Galilean space-time. Thus although absolute motion cannot be measured using light and matter waves, there is no reason to presume that absolute motion has no intrinsic meaning. Indeed, if another type of wave could be measured (e.g. gravity waves) then it may be possible to determine absolute motion with respect to the aether. The interpretation of relativity as a physical property of space-time is a philosophical preference that is in no way justified by evidence.

It is often emphasized that absolute motion cannot be determined. This claim is actually doubtful, since motion relative to the cosmic background microwave radiation *can* be determined. More importantly, it is possible to determine absolute acceleration. Two observers undergoing a change in relative velocity can determine which of them is accelerating because only the accelerating observer will experience a force. If the inertial (constant velocity) observer sees that an accelerated object has changed its length and clock rate, he can reasonably conclude that the acceleration caused real changes to the object. Consistency therefore demands that the accelerated observer should attribute any observed changes in length and clock rate of distant objects to changes in his own accelerated rulers and clocks.

Special relativity is entirely consistent with the ordinary limitations of measurement in a Euclidean space with absolute time. This simple fact explains why classical models of disturbances in the aether have historically produced physical equations consistent with the Principle of Relativity.

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Figures

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Figure 2.2 Thomas Young (1773 – 1829)
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Figure 2.3 Augustin Fresnel (1788 – 1827).
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Figure 2.4 George Gabriel Stokes (1819 – 1903).
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Figure 2.13 Neils Bohr (1885-1962).
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Chapter 3. Elastic Waves and Quantum Mechanics

"An ocean traveler has even more vividly the impression that the ocean is made of waves than that it is made of water."
—Arthur S. Eddington

3.1. Introduction

"... we must hang on to the basic ideas of logic at all costs."
— Paul Adrian Maurice Dirac [1989]

The theoretical developments discussed in this book were accompanied by myriad experimental discoveries, most notably in the laboratories of J. J. Thomson [**Figure 3.1**] and his student (and later successor at Cambridge) Ernest Rutherford [**Figure 3.2**]. J.J. Thomson's study of cathode rays led to his discovery of the electron [1897]. Rutherford [1911, 1914] observed that beams of alpha particles occasionally scatter at large angles from a thin target. This observation led him to propose that atoms contain a positively charged nucleus of extremely small size (of order 10^{-12} cm radius) surrounded by a much larger volume (of order 10^{-8} cm radius) of negatively charged electrons. The Rutherford atomic model became the basis for all future theories of atomic structure.

Figure 3.1 .J. Thomson (1856-1940)

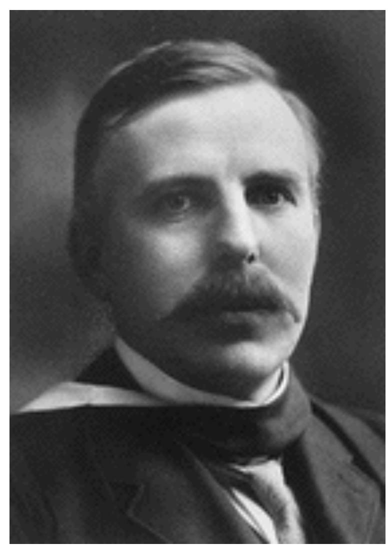

Figure 3.2 Ernest Rutherford (1871-1937)

We have already mentioned the beginnings of quantum theory in the introduction to the previous chapter. Now we will discuss events which led to the development of a wave equation for the electron. This synopsis is based largely on Whittaker [1954].

According to Bohr's atomic model [Bohr 1913] the electron energy levels in hydrogen are:

$$W = -\frac{2\pi^2 e^4 m}{h^2 n^2} = -\frac{e^4 m}{2\hbar^2 n^2} = -R\frac{1}{n^2} \tag{3-1}$$

where R is called the Rydberg constant and $\hbar = h/2\pi$. Radiation is emitted when an electron drops from a higher energy level (larger n) to a lower energy level (smaller n), and the frequency of the radiation is proportional to the difference in energies.

Figure 3.3 Arnold J.W. Sommerfeld (1868-1951)

William Wilson [1915] and Arnold J. W. Sommerfeld [1915a, 1915b, 1916a] [**Figure 3.3**] recognized Bohr's quantization of angular momentum of circular orbits (yielding energy quantum number n) to be a special case of quantization of action: $\oint p_i dq_i = h$, where q_i is a coordinate variable and p_i is the corresponding momentum. Sommerfeld explained much of the 'fine structure' of hydrogen spectral lines by generalizing Bohr's circular orbits to ellipses, including relativistic inertia corrections and a new azimuthal quantum number k. The relativistic correction to the energy levels of hydrogen-like atoms is:

$$\Delta W = \frac{e^4}{c^2\hbar^2} R \frac{Z^4}{n^3 k(k-1)} \tag{3-2}$$

The fine structure constant, $e^2/c\hbar \approx 1/137$, represents the ratio between the velocity of the first Bohr orbit and the speed of light [Whittaker 1954, p. 120].

Karl Schwarzchild [1916] and Paus Sophus Epstein [1916] used action quantization to derive the spectral line shifts for hydrogen in a strong electric field (Stark effect). Sommerfeld

[1916b] and Peter Debye [1916] explained the splitting of spectral lines in a strong magnetic field (Zeeman effect) by using three quantization conditions: energy (*n*), magnitude of orbital angular momentum ($l = k - 1 \le n$), and component of angular momentum parallel to the applied magnetic field (*m*). Note that $|m| \le l$. Quantization of a single component of angular momentum, termed 'space quantization', was verified when O. Stern and W. Gerlach [1921] split a beam of silver atoms into two discrete components simply by applying a nonuniform magnetic field.

Principal spectral lines of alkali elements (e.g. Na) are doublets which could not be explained by the aforementioned quantum numbers. Various schemes were proposed to include an additional angular momentum quantum number which was generally supposed to be associated with the atomic core. Wolfgang Pauli [**Figure 3.4**] disputed this identification of core angular momentum in part because it led to a Z^3 dependence in the relativistic energy shifts. He instead attributed the quantum number *j* to the radiant electron which possessed a "classically non-describable two-valuedness". Pauli [1925] also observed that restriction of each set of quantum numbers *n*, *k*, *j*, and *m* to a single electron (the 'exclusion principle') was consistent with the notion of electron shells (proposed by Edmund C. Stoner and J. D. Main Smith) which close when all of the quantum numbers for a given value of *n* are filled by electrons.

Ralph Kronig realized that self-rotation of the electron with angular momentum of $\hbar/2$ would explain the Z^4-dependence of the doublet energy shifts, but since his calculation of the energy levels was off by a factor of two he did not publish his idea. Uhlenbeck and Goudsmidt [1925] did publish the idea of electron angular momentum of $\hbar/2$, but unsuccessfully attempted to withdraw the paper after realizing the factor of two discrepancy. At this time Llewellyn Hilleth Thomas [1926, 1927] resolved the factor of two discrepancy by publishing a paper which demonstrated that the (classical) relativistic precession of the electron magnetic moment in the internal atomic magnetic field, and hence the splitting of energy levels, had been computed incorrectly. Hence the electron's spin angular momentum of $\hbar/2$ was established.

Figure 3.4 Wolfgang Pauli (1900-1958)

Figure 3.5 Werner Heisenberg (1901-1976)

Werner Heisenberg [1925] [**Figure 3.5**] proposed that transitions between stationary states (e.g. *m* and *n*) could be represented by an array of elements (e.g. x_{mn}) whose amplitude is related to the likelihood of the transition. Max Born [1925] and Pascual Jordan quickly developed this idea into a complete formulation of matrix mechanics in which commutation rules replaced action integrals as the basis of quantization (e.g. $qp - pq = \mathrm{i}\hbar$ where q is a coordinate and p is the conjugate momentum).

Louis de Broglie [1924] proposed a novel explanation for Bohr's quantization rules. He proposed that matter has a wavelike character with energy proportional to frequency $\varepsilon = \hbar\omega$ and momentum proportional to wave vector $\mathbf{p} = \hbar\mathbf{k}$. The periodic condition for a wave of wavelength λ propagating in a circular orbit of radius *r*:

$$2\pi r = n\lambda \tag{3-3}$$

implies quantization of angular momentum:

$$rp = n\hbar \tag{3-4}$$

Figure 3.6 Erwin Schrodinger (1887-1961)

Erwin Schrödinger [1926] [**Figure 3.6**] subsequently published a differential wave equation based on de Broglie's matter waves. For a non-relativistic particle of mass *m* in a potential $V(\mathbf{r},t)$, the energy is given by:

$$E = \frac{p^2}{2m} + V \tag{3-5}$$

The corresponding differential equation for de Broglie waves is called the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{3-6}$$

where the wave function ψ is a complex scalar. For a Coulomb potential ($V = -e^2/r$) this equation yields energy eigenvalues equal to Bohr's energy levels. Schrödinger initially interpreted the wave function to be related to electrical charge density, but Max Born's [1926] interpretation of $\psi^*\psi$ as a probability density was soon widely accepted. A probability conservation equation can be obtained by multiplying ψ^* and adding the complex conjugate:

$$\frac{\partial}{\partial t}|\psi|^2 + -\frac{\mathrm{i}\hbar}{2m}\left\{\psi^*\left[\nabla^2\psi\right] - \left[\nabla^2\psi^*\right]\psi\right\} \tag{3-7}$$

The Schrödinger equation has the classical Hamiltonian form (see e.g. Goldstein [1980]):

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial t} + H\psi = 0 \tag{3-8}$$

with $-\mathrm{i}\hbar\psi$ representing Hamilton's principal function whose gradient is the momentum $\mathbf{p}$.

The differential equation corresponding to the relativistic energy-momentum relation $E^2 = p^2c^2 + m_0^2c^4$ is called the Klein-Gordon equation (or relativistic Schrödinger equation):

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2c^2\nabla^2\psi + m_0^2c^4\psi \tag{3-9}$$

Interpretation of this equation proved more difficult than Schrödinger's non-relativistic equation. It does not have the classical Hamiltonian form with a first-order time derivative. The resulting conservation equation is obtained by multiplying ψ^* and subtracting the complex conjugate:

$$\frac{\partial}{\partial t}\left\{\frac{\mathrm{i}\hbar e}{2mc^2}\left[\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right]\right\} + \nabla\cdot\left\{\frac{-\mathrm{i}\hbar e}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]\right\} = 0 \tag{3-10}$$

The density in this equation (the first square brackets) can have either sign, making it problematic as an expression for probability density. Nonetheless the Klein-Gordon equation eventually became accepted as a description of particles with zero spin.

Schrödinger subsequently demonstrated that Heisenberg's commutation rule $qp - pq = \mathrm{i}\hbar$ follows immediately from the definition of conjugate momenta as derivatives:

$$q\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right)\psi - \left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right)q\psi = \mathrm{i}\hbar\psi \tag{3-11}$$

Pauli [1927] multiplied Schrodinger's wave function by a two-component factor (termed a *spinor*) to model the two-valued space quantization due to electron spin. Multiplicative operators on Pauli spinors are linear combinations of independent 2×2 matrices which by convention are:

$$I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3-12}$$

The last three of these matrices form a vector (i.e. transform as a vector under rotations) and are called the Pauli matrices.

Figure 3.7 Paul Dirac (1902-1984)

Paul Dirac [1928] [**Figure 3.7**] finally derived a valid relativistic wave equation by extending the wave function to four components and using matrix coefficients. The Dirac wave function has four complex components which can be written as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \end{bmatrix}^{\mathrm{T}} \tag{3-13}$$

Such a wave function is called a Dirac spinor or bispinor. A Dirac spinor can be decomposed into left- and right-handed Pauli spinors which each have two complex components. Dirac's equation describing an electron in an electromagnetic potential is:

$$i\hbar \frac{\partial}{\partial t}\psi = -i\hbar c\boldsymbol{\alpha} \cdot \nabla \psi + \beta m_e c^2 \psi + \left(e\Phi - e\boldsymbol{\alpha} \cdot \mathbf{A}\right)\psi \tag{3-14}$$

where $\boldsymbol{\alpha}$ and β are the matrices:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \quad \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3-15}$$

Dirac also demonstrated that quantum mechanical equations could describe multiple particles by introducing a new wave function whose integrated square magnitude is taken to be the number of particles. This procedure is called "second quantization" (see e.g. [Tomonaga 1974]). Dirac developed this method for bosons by assuming the scalar amplitudes (a_k) of various states (*k*) to be operators which satisfy the commutation relation $a_k a_l^\dagger - a_k^\dagger a_l = \delta_{kl}$. The product $a_k^\dagger a_k$ then has non-negative integer eigenvalues and represents the number of particles in each state. Jordan and Eugene Wigner [1928] adapted this idea to fermions by using an anti-commutation relation $a_k a_l^\dagger + a_k^\dagger a_l = \delta_{kl}$. In this case the product $a_k^\dagger a_k$ has eigenvalues of zero and one, consistent with Pauli's exclusion principle.

Dirac's research led him to believe in the existence of an aether: "If one examines the question in light of present-day knowledge, one finds that the aether is no longer ruled out by relativity, and good reasons can now be advanced for postulating an aether…." [Dirac 1951]. His rationale was that an aether velocity was required for setting up a Hamiltonian formulation of the action principle.

Dirac's equation remains the foundation for describing matter waves. The Standard Model of particle physics "asserts that the material in the universe is made up of elementary fermions interacting through fields, of which they are the sources. The particles associated with the interaction fields are bosons." [Cottingham and Greenwood 1998]. The wave functions are regarded as dimensionless quantities whose magnitude at any point represents a probability density for the presence of one or more particles. Some efforts were made to formulate a classical interpretation of the wave function (notably by de Broglie [1928] and David Bohm [1952], see e.g. Goldstein [2002]) but none was successful in the 20th century.

The mathematical and geometrical properties of spinors were first studied by the mathematician Élie Cartan in 1913 (see e.g. Hladik [1999] for a mathematical analysis of spinors). The algebra of spinors is closely related to that of quaternions, which were invented by Sir William R. Hamilton around 1843 as a generalization of complex numbers to higher dimension. Quaternions consist of four real components. They can in fact be written in matrix form with basis vectors *I*, σ_x, σ_y, and σ_z.

Spinors have historically been regarded by mathematicians as operators (linear representations of rotation groups) and by physicists as abstract quantities with no classical interpretation. However, David Hestenes [1967] developed a space-time algebra which provides a geometrical interpretation of the Dirac equation. The wave function describes a generalized Lorentz rotation (spatial rotation and velocity boost) in addition to an amplitude and one additional parameter which appears to transform between matter and anti-matter.

There have been successful attempts to reformulate the Dirac theory in terms of relations between local physical observables [Takabayashi 1957, Hestenes 1973]. The Dirac equation uniquely determines the evolution of local dynamical quantities such as angular momentum density, linear momentum density, and energy density. In other words the Dirac equation is deterministic with respect to dynamical quantities.

In this chapter we will derive a Dirac equation to describe rotational waves in an elastic solid. We will regard 'particles' as soliton solutions. We will then derive numerous properties of elementary particles from this model.

3.2. Torsion Waves

...there are circumstances in which mathematics will produce results which no one has really been able to understand in any direct fashion. An example is the Dirac equation, which appears in a very simple and beautiful form, but whose consequences are hard to understand.
— Richard P. Feynman, Robert B. Leighton, and Matthew Sands [1963a]

Quantum theory developed from an initial classical picture of matter as particles. Yet we have seen that special relativity is a natural consequence of the wave nature of matter. Therefore the classical theory which corresponds to quantum mechanics must be a wave theory. One historical dilemma of quantum wave theory is the lack of an obvious physical interpretation of the wave amplitudes. Max Born suggested that the wave intensity be interpreted as a probability density, but he emphasized that "...the probability itself is propagated in accordance with the law of causality" [Born 1926]. While there is no doubt that the quantum wave functions can predict the likelihood of experimental results, their evolution indicates causal rather than stochastic interactions.

Actually, the dynamical interpretation of the wave functions can be resolved by simple dimensional analysis. In terms of Dirac spinors, the z-component of spin angular momentum density s_z is:

$$s_z = \left[\frac{\hbar}{\int d^3r\, |\psi|^2}\right]\frac{1}{2}\psi^\dagger\sigma_z\psi \tag{3-16}$$

where ψ is the 4-component complex wave function with $|\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ and σ_z is the z-component spin angular momentum matrix:

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3-17}$$

The leading factor in Eq. 3-16 is simply a constant which establishes units.

Construction of a classical wave theory of matter must therefore begin with waves carrying angular momentum. Classically, angular momentum is associated with rotations of inertial bodies. Waves of angular momentum require not only inertia but also torque which resists rotations. Generation of torque in response to local rotations implies elasticity. Therefore the classical model of matter waves consists of rotations in an elastic solid (torsion or shear waves). We already know that the elastic solid was the basis for classical wave theories of light, so we can proceed with some confidence.

First consider torsion in one dimension, such as on a torsion wave machine or a stretched-out rubber band [Figure 3.8]. A torsion wave machine has at least one intriguing parallel with particle physics. If one rotates a single rod near the center of the wire, a right-handed twist propagates in one direction and a left-handed twist propagates in the other direction, analogous to the production of particles and anti-particles. In every known physical process, anti-matter behaves like a mirror image of matter. Another interesting property of 1-D rotations is that there is a natural distinction between rotations of odd and even multiples of π, analogous to the distinction between odd (fermions) and even (bosons) multiples of the unit angular momentum $\hbar/2$. The notion that torsion should be associated with matter is in fact widely accepted [Kleinert 1989].Therefore there is reason to believe that a mathematical analysis of torsion waves might provide some clues to the interpretation of quantum mechanics. This analogy was explored by Close [2002].

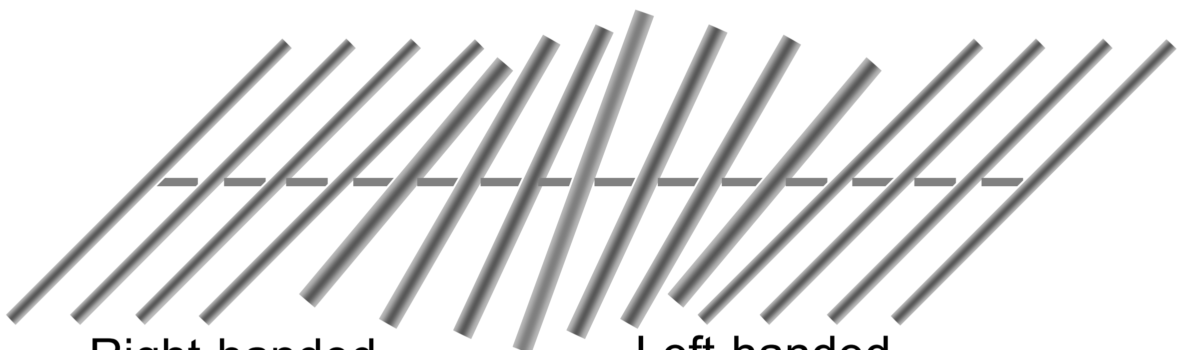

Figure 3.8 Rotation of a single bar on a torsion wave machine results in mirror-symmetric waves propagating in opposite directions. This is a one-dimensional analogue of production of particles and anti-particles. Matter and anti-matter are similarly produced in pairs, and behave physically as mirror images of one another.

If the moment of inertia per unit length is *I*, and the torsion spring constant of the wire (or rubber band) is *K*, then the wave equation is given by:

$$I\frac{\partial^2\Theta(z,t)}{\partial t^2}=K\frac{\partial^2\Theta(z,t)}{\partial z^2} \tag{3-18}$$

where $\Theta(z,t)$ is the orientation at axial position *z* and time *t*. The wave speed is given by $c=\sqrt{K/I}$.

As with displacement waves, a unique frequency and wavelength cannot be defined for torsion waves unless many cycles are produced in succession. If one end of the wave machine is rotated at a constant rate ω, the torsion waves propagate along the machine with uniform wavelength $\lambda = c/\omega$. Each rod along the machine rotates with the constant driving frequency ω. The angular momentum per unit length ℓ is therefore $\ell = I\omega = Ick = Ic\,\partial\Theta/\partial z$. The angular momentum is therefore proportional to the spatial derivative of the angle. The angular momentum of a twist from 0 to Θ_0 can be obtained by integrating over angle:

$$L_0 = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} I\frac{d\Theta}{dt}dz = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} Ic\frac{d\Theta}{dz}dz = \int_0^{\Theta_0} Icd\Theta = Ic\Theta_0 \tag{3-19}$$

Thus we see that the total angular momentum of a twist is proportional to the rotation angle and independent of frequency.

A twist propagating with constant wavelength has no torque, so the kinetic and potential energies remain constant as the wave propagates. The kinetic energy per unit length is $I\omega^2/2$ and the potential energy per unit length is $\frac{1}{2}K(\partial\Theta/\partial z)^2 = 2\pi^2 K/\lambda^2 = I\omega^2/2$. Integration from 0 to Θ_0 yields for the total energy:

$$\varepsilon = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} I\omega\left|\frac{d\Theta}{dt}\right|dz = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} Ic\omega\left|\frac{d\Theta}{dz}\right|dz = \int_0^{\Theta_0} Ic\omega d\Theta = Ic\omega\Theta_0 = L_0\omega \tag{3-20}$$

The wave energy is equal to the wave angular momentum times the angular frequency. This is analogous to the energy quantum of $\hbar\omega$. At this point we make the identifications:

$$I = \frac{L_0}{c\Theta_0}$$
$$K = c^2 I = \frac{L_0 c}{\Theta_0} \tag{3-21}$$

so that the wave equation is simply:

$$\frac{\partial^2\Theta(z,t)}{\partial t^2} = c^2\frac{\partial^2\Theta(z,t)}{\partial z^2} \tag{3-22}$$

Incidentally, although we have been describing torsion waves along a thin wire, the equation is valid for torsion waves in a thick cylindrical rod (see e.g. Feynman et al. [1963b]).

The case of a thin elastic rod has been studied by Matsutani and Tsuru [1992], who interpreted nonlinear waves as fermions. We will also arrive at a fermionic interpretation of nonlinear waves when we study an infinite 3-D elastic solid below.

Now we will take a look at the classical wave equation to see if it can be applied to the study of matter. We will start with one-dimensional waves as above, then generalize to three dimensional scalar and vector waves.

3.3. One-Dimensional Scalar Waves

"I have deep faith that the principle of the universe will be beautiful and simple."
—Albert Einstein

Consider a scalar quantity (a) which satisfies a wave equation with wave speed (c) in one spatial dimension (z):

$$\partial_t^2 a = c^2 \partial_z^2 a \tag{3-23}$$

This equation can be factored:

$$[\partial_t + c\partial_z][\partial_t - c\partial_z]a = 0 \tag{3-24}$$

The general solution is a superposition of forward (a_{F}) and backward (a_{B}) propagating waves:

$$a(z,t) = a_{\mathrm{F}}(z-ct) + a_{\mathrm{B}}(z+ct) \tag{3-25}$$

This form of the solution to the one-dimensional wave equation can be found in any elementary textbook on waves. We can write the equations for forward and backward waves in matrix form:

$$\left[\partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c\partial_z \right]\begin{pmatrix} a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct) \end{pmatrix} = 0 \tag{3-26}$$

The spatial derivatives are related to the temporal derivatives:

$$c\partial_z \begin{pmatrix} a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct) \end{pmatrix} = \partial_t \begin{pmatrix} -a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct) \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_t \begin{pmatrix} a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct) \end{pmatrix} \tag{3-27}$$

Let $\dot{a} \equiv \partial_t a$ and $a' \equiv \partial_z a$. We now define a wave function in terms of the time derivatives:

$$\mathbf{\Psi} \equiv \begin{pmatrix} \dot{a}_{\mathrm{F}}(z-ct) \\ \dot{a}_{\mathrm{B}}(z+ct) \end{pmatrix} \tag{3-28}$$

The wave equation for the forward and backward waves is now:

$$\left[\partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c\partial_z \right]\mathbf{\Psi} = \partial_t \begin{pmatrix} \dot{a}_{\mathrm{F}}(z-ct) \\ \dot{a}_{\mathrm{B}}(z+ct) \end{pmatrix} - c\partial_z \begin{pmatrix} a'_F(z-ct) \\ a'_B(z+ct) \end{pmatrix} = 0 \tag{3-29}$$

We have now reduced the second-order wave equation to a first-order matrix equation.

3.3.1. Spinors and Bispinors

If we regard the z-axis as one of three orthogonal axes, then the two independent components $\dot{a}_{\mathrm{F}}$ and $\dot{a}_{\mathrm{B}}$ differ by a 180 degree rotation. This is the definitive property of independent states in spin one-half systems. Unfortunately, this property is de-emphasized (or even unrecognized) in the physics literature in favor of the more exotic property that complex spinors change sign upon 360 degree rotation. This latter property does not apply to physical observables which are computed from bilinear products of spinors. However, the separation of independent states by 180 degrees does apply to wave velocity, implying that solutions of the wave equation generally form spin one-half systems. Note that unlike positive and negative scalars or vector components (which can also be expressed as bilinear products of spinors), waves with positive and negative velocity are not related by a multiplicative factor of minus one. The forward and backward waves are independent states [Figure 3.9]. The mathematical basis of this property is that wave velocity is a property of the functional arguments and is not simply an amplitude.

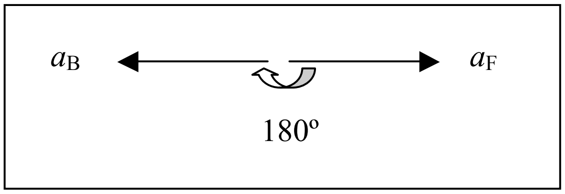

Figure 3.9 Waves propagating in opposite directions along an axis comprise independent states separated by a 180° rotation. This is the basis of half-integer spin.

The relationship between waves and spinors can be made explicit as in Close (2002) by further decomposition into positive-definite components $(\dot{a}_{F+}, \dot{a}_{B+}, \dot{a}_{F-}, \dot{a}_{B-})$ or $(a'_{F+}, a'_{B+}, a'_{F-}, a'_{B-})$ representing positive (+) or negative (−) contributions to the wave derivatives:

$$\dot{a}(z,t) = \dot{a}_{F+}(z-ct) - \dot{a}_{F-}(z-ct) + \dot{a}_{B+}(z+ct) - \dot{a}_{B-}(z+ct) \tag{3-30}$$

and

$$\begin{aligned} ca'(z,t) &= c\left[a'_{F+}(z-ct) - a'_{F-}(z-ct) + a'_{B+}(z+ct) - a'_{B-}(z+ct)\right] \\ &= -\dot{a}_{F+}(z-ct) + \dot{a}_{F-}(z-ct) + \dot{a}_{B+}(z+ct) - \dot{a}_{B-}(z+ct) \end{aligned} \tag{3-31}$$

From here on the functional arguments will not be written explicitly. Note that the positive-definite components may have discontinuous derivatives where the original signed quantities pass continuously through zero. For example, to make the time derivatives continuous requires matching conditions for $\dot{a}$:

$$\begin{aligned} \partial_t \dot{a}_{F+}\big|_{\dot{a}_{F+}=\dot{a}_{F-}=0} &= -\partial_t \dot{a}_{F-}\big|_{\dot{a}_{F+}=\dot{a}_{F-}=0} \\ \partial_z \dot{a}_{F+}\big|_{\dot{a}_{F+}=\dot{a}_{F-}=0} &= -\partial_z \dot{a}_{F-}\big|_{\dot{a}_{F+}=\dot{a}_{F-}=0} \end{aligned} \tag{3-32}$$

Similar relations hold for the backward wave components. Such discontinuities do not affect the validity of the first order equations. However, higher derivatives may be undefined at some points.

Since each component has a unique sign, we can express $\dot{a}$ and a' in spinorial form with the one-dimensional wave function ψ_v (the subscript 'v' refers to the velocity axis):

$$\dot{a} = \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv \psi_v^T \sigma \psi_v$$

$$ca' = \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv -\psi_{\mathrm{v}}^T \beta \psi_{\mathrm{v}} \qquad (3\text{-}33)$$

where the superscript T indicates transposition of the column matrix and the matrix $\beta\sigma$ tabulates the forward and backward velocities (v):

$$v\psi_{\mathrm{v}} = c \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv c\beta\sigma\psi_{\mathrm{v}} \qquad (3\text{-}34)$$

This wave function is a one-dimensional bispinor. In one dimension the components of the bispinor may be taken to be real and positive-definite. Extension to three dimensions requires complex components.

Changing the order of terms in the wave function is called a change of 'representation'. A few important points are:

1. The components of the column matrix wave function are real and positive-definite.

2. Only one forward component and one backward component can be non-zero at any given time and place (for one-dimensional waves).

3. The spatio-temporal variation of each component must be consistent with its location in the column matrix.

Since some of the components must be zero, let δ_{F} and δ_B be either zero or one. Then the wave function is:

$$\psi_{\mathrm{v}} = \left[\dot{a}_{\mathrm{F}}^{1/2}\delta_{\mathrm{F}} \quad \dot{a}_{\mathrm{B}}^{1/2}\delta_B \quad \dot{a}_{\mathrm{B}}^{1/2}[1-\delta_{\mathrm{B}}] \quad \dot{a}_{\mathrm{F}}^{1/2}[1-\delta_{\mathrm{F}}]\right]^{\mathrm{T}} \qquad (3\text{-}35)$$

Using Lorentz boosts, the wave function can be written as:

$$\psi_{\mathrm{v}} = \dot{a}_0^{1/2} \exp(\beta\sigma\alpha/2)\left[\delta_{\mathrm{F}} \quad \delta_B \quad [1-\delta_B] \quad [1-\delta_{\mathrm{F}}]\right]^{\mathrm{T}} / \sqrt{2} \qquad (3\text{-}36)$$

This form has two independent continuous parameters and two binary parameters.

The equation of evolution of the wave components is:

$$\partial_t\psi_{\mathrm{v}} + c\beta\sigma\,\partial_z\psi_{\mathrm{v}} = 0 \qquad (3\text{-}37)$$

This is the one-dimensional Dirac equation. This equation can be interpreted as a convective derivative with two opposite velocities represented by the matrix $v=c\beta\sigma$.

The relation between one dimensional bispinor equations and scalar wave equations is summarized in **Table 3-I**.

Table 3-I Corresponding Bispinor and Scalar Wave Equations in One Dimension

Bispinor Equation	Scalar Equation
$\partial_t\left[\psi_v^T\sigma\psi_v\right]+c\partial_z\left[\psi_v^T\beta\psi_v\right]=0$	$\partial_t^2 a-c^2\partial_z^2 a=0$
$\partial_t\left[\psi_v^T\psi_v\right]+c\partial_z\left[\psi_v^T\beta\sigma\psi_v\right]=0$	$\partial_t\lvert\partial_t a_F\rvert+\partial_t\lvert\partial_t a_B\rvert+c^2\partial_z\lvert\partial_z a_F\rvert-c^2\partial_z\lvert\partial_z a_B\rvert=0$
$\partial_t\left[\psi_v^T\beta\psi_v\right]+c\partial_z\left[\psi_v^T\sigma\psi_v\right]=0$	$\partial_t\left[-c\partial_z a\right]+c\partial_z\left[\partial_t a\right]=0$
$\partial_t\left[\psi_v^T\beta\sigma\psi_v\right]+c\partial_z\left[\psi_v^T\psi_v\right]=0$	$c\partial_t\lvert\partial_z a_F\rvert-c\partial_t\lvert\partial_z a_B\rvert+c\partial_z\lvert\partial_t a_F\rvert+c\partial_z\lvert\partial_t a_B\rvert=0$

3.3.2. Wave Velocity

The mean velocity (v) of the wave is proportional to the ratio between the difference and sum of the forward and backward components [Close 2002]:

$$v=c\frac{\lvert\dot{a}_F\rvert-\lvert\dot{a}_B\rvert}{\lvert\dot{a}_F\rvert+\lvert\dot{a}_B\rvert}=c\frac{\psi_v^T\beta\sigma\psi_v}{\psi_v^T\psi_v} \tag{3-38}$$

Since $\lvert\dot{a}_F\rvert$ and $\lvert\dot{a}_B\rvert$ are positive-definite, we can define them by the relation:

$$\begin{aligned}\lvert\dot{a}_F\rvert&=\dot{a}_0\exp(\alpha)\\ \lvert\dot{a}_B\rvert&=\dot{a}_0\exp(-\alpha)\end{aligned} \tag{3-39}$$

so that our definition of velocity is:

$$v=c\frac{\dot{a}_0\exp(\alpha)-\dot{a}_0\exp(-\alpha)}{\dot{a}_0\exp(\alpha)+\dot{a}_0\exp(-\alpha)}=c\tanh\alpha \tag{3-40}$$

If we start from a zero-velocity state with $\lvert\dot{a}_\mathrm{F}\rvert=\lvert\dot{a}_\mathrm{B}\rvert=\dot{a}_0$, then we can change the velocity using the 'Lorentz boost' operator ($\psi_v\rightarrow\exp(\beta\sigma\alpha/2)\psi_v$):

$$v=\frac{\left[\psi_v^T\exp(\beta\sigma\alpha/2)\right]c\beta\sigma\left[\exp(\beta\sigma\alpha/2)\psi_v\right]}{\left[\psi_v^T\exp(\beta\sigma\alpha/2)\right]\left[\exp(\beta\sigma\alpha/2)\psi_v\right]}=c\frac{\exp(\alpha)-\exp(-\alpha)}{\exp(\alpha)+\exp(-\alpha)}=c\tanh\alpha \tag{3-41}$$

Note that successive boosts preserve the form of the operator:

$$\exp(\beta\sigma\alpha_2/2)\exp(\beta\sigma\alpha_1/2)=\exp(\beta\sigma[\alpha_1+\alpha_2]/2) \tag{3-42}$$

This property enables us to recover the relativistic equation for addition of parallel velocities:

$$v_1 = c \tanh \alpha_1$$
$$v_2 = c \tanh \alpha_2$$
$$v = c \tanh(\alpha_1 + \alpha_2) = c \frac{\tanh \alpha_1 + \tanh \alpha_2}{1 + \tanh \alpha_1 \tanh \alpha_2} = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2} \tag{3-43}$$

This result is another example of how the laws of special relativity apply to classical waves in ordinary Galilean space-time, as discussed in Chapter 2.

Using Lorentz boosts, the wave function can be written as:

$$\psi_{\mathrm{v}} = \frac{1}{\sqrt{2}} \dot{a}_0^{1/2} \exp(\beta\sigma\,\alpha/2)\left[\delta_{\mathrm{F}} \quad \delta_B \quad [1-\delta_{\mathrm{F}}] \quad [1-\delta_B]\right]^T \tag{3-44}$$

This form has two independent continuous parameters and two binary parameters.

3.4. Three Dimensional Scalar Waves

"... in quantum phenomena one obtains quantum numbers, which are rarely found in mechanics but occur very frequently in wave phenomena and in all problems dealing with wave motion."
— Louis de Broglie [1963]

3.4.1. Rotation of Gradient and Velocity

The spatial derivative ∂_z generalizes in three dimensions to a arbitrary direction ∂_{v}, where the index (v) represents an arbitrary direction. Wave velocity is defined to be parallel to the gradient. Since the matrix $\beta\sigma$ is associated with a particular axis, it must be one component of a vector. We can let the matrix $\beta = \beta_3$ and define the gradient matrix components as:

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & -\tilde{\mathrm{i}} & 0 \\ 0 & 0 & 0 & -\tilde{\mathrm{i}} \\ \tilde{\mathrm{i}} & 0 & 0 & 0 \\ 0 & \tilde{\mathrm{i}} & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3-45}$$

The symbol ($\tilde{\mathrm{i}}$) represents a unit pseudoscalar imaginary which is odd (changes sign) with respect to spatial inversion. This property is necessary because velocity is a polar vector and:

$$\tilde{\mathrm{i}} = \beta_1 \beta_2 \beta_3 \tag{3-46}$$

We must now allow the wave function to have complex components. These matrices have commutation relations equivalent to the Pauli matrices:

$$\beta_i \beta_j + \beta_j \beta_i = 2\delta_{ij} \; ; \; \beta_i \beta_j - \beta_j \beta_i = 2\,\tilde{\mathrm{i}}\,\varepsilon_{ijk} \beta_k \tag{3-47}$$

An elegant way to write these commutation relations is:

$$\beta_i \beta_j = \beta_i \cdot \beta_j + \tilde{\mathrm{i}}\, \beta_i \times \beta_j \tag{3-48}$$

where:

$$\beta_i \cdot \beta_j = \frac{1}{2}\left[\beta_i\beta_j + \beta_j\beta_i\right]$$
$$\beta_i \times \beta_j = -\frac{\tilde{\mathrm{i}}}{2}\left[\beta_i\beta_j - \beta_j\beta_i\right]$$

(3-49)

Hence we can regard these matrices as basis vectors whose commutation relations express their relative orientation. This idea is the basis for the mathematical field of *geometric algebra*. Notice that the unit imaginary now has a geometrical interpretation as the product of three orthogonal unit vectors (i.e. an oriented unit volume):

$$\beta_1\beta_2\beta_3 = \beta_1\left[\beta_2 \cdot \beta_3 + \tilde{\mathrm{i}}\,\beta_2 \times \beta_3\right] = \tilde{\mathrm{i}}\,\beta_1 \cdot \left[\beta_2 \times \beta_3\right] = \tilde{\mathrm{i}} \qquad (3\text{-}50)$$

The rotation operators for this space have the form:

$$R_{\beta_j}\left(\zeta_i\right) = \exp\left(-\tilde{\mathrm{i}}\,\beta_i\zeta_i/2\right)\beta_j \exp\left(\tilde{\mathrm{i}}\,\beta_i\zeta_i/2\right) = \beta_j \cos\zeta_i - \frac{\tilde{\mathrm{i}}}{2}\left[\beta_i\beta_j - \beta_j\beta_i\right]\sin\zeta_i \qquad (3\text{-}51)$$

which can be written in vector form:

$$R_{\boldsymbol{\sigma}}\left(\boldsymbol{\zeta}\right) = \exp\left(-\tilde{\mathrm{i}}\,\beta_\zeta\zeta/2\right)\boldsymbol{\beta}\exp\left(\tilde{\mathrm{i}}\,\beta_\zeta\zeta/2\right) = \boldsymbol{\beta}\cos\zeta + \left[\boldsymbol{\beta}_\zeta \times \boldsymbol{\beta}\right]\sin\zeta_i \qquad (3\text{-}52)$$

To include rotations, the one-dimensional derivative $c\partial_{\mathrm{v}}a = -\psi_{\mathrm{v}}^T\beta_{\mathrm{v}}\psi_{\mathrm{v}}$ must be modified to include orientation. This orientation is computed relative to the x_3-axis. Using the definitions:

$$\beta_{\mathrm{v}} = \left[\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\beta_3 \exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\right]$$
$$\psi = \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_{\mathrm{v}}$$
$$\psi^\dagger = \psi_{\mathrm{v}}^T \exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)$$

(3-53)

The wave function now has complex components. The rotation operator $R_{\boldsymbol{\zeta}}\left(\psi_{\mathrm{v}}\right) = \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_{\mathrm{v}}$ applied to the one-dimensional wave function inverts the rotation of the basis vectors so that the derivative can be evaluated using the one-dimensional real-valued matrix β_3 and wave function ψ_{v}.

The spatial derivative is:

$$c\partial_{\mathrm{v}}a = -\psi_{\mathrm{v}}^T\beta_3\psi_{\mathrm{v}} = -\psi^\dagger\beta_{\mathrm{v}}\psi \qquad (3\text{-}54)$$

Since the beta matrices are mutually orthogonal, the components of $\psi^\dagger\boldsymbol{\beta}\psi$ perpendicular to x_{v} must be zero. Therefore the three dimensional gradient is:

$$c\nabla a = -\hat{e}_{\mathrm{v}}\psi_{\mathrm{v}}^T\beta_3\psi_{\mathrm{v}} = -\psi^\dagger\boldsymbol{\beta}\psi \qquad (3\text{-}55)$$

3.4.2. *Successive Rotations*

Successive rotations can be performed using either fixed axes or embedded axes. The result of successive rotations about fixed axes depends on the order in which the rotations are taken. For example, successive rotations of $\pi/2$ about the x_1- and x_2-axes move e_3 to either $-e_2$ or $+e_1$, depending on the order. Hence:

$$\exp\left(-\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\left[\exp\left(-\tilde{\mathrm{i}}\,\beta_1\pi/4\right)\beta_3\exp\left(\tilde{\mathrm{i}}\,\beta_1\pi/4/2\right)\right]\exp\left(\tilde{\mathrm{i}}\,\beta_2\pi/4\right)=-\beta_2$$
$$\exp\left(-\tilde{\mathrm{i}}\,\beta_1\pi/4\right)\left[\exp\left(-\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\beta_3\exp\left(\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\right]\exp\left(\tilde{\mathrm{i}}\,\beta_1\pi/4/2\right)=\beta_1 \tag{3-56}$$

Here the expression inside the square brackets is evaluated first, followed by applying the rotation operator outside the square brackets. If we interpret these rotation operators as acting on spinors then the order appears to be backward. The expression: $\exp\left(\tilde{\mathrm{i}}\,\beta_1\pi/4/2\right)\exp\left(\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\psi$ represents spinor rotation of $-\pi/2$ about the x_1-axis followed by rotation about the x_2-axis.

3.4.3. *Euler Angles*

We can put the operations back in order if we consider the second rotation operator to have been rotated along with the wave function by the first one:

$$R'(\mathbf{\Theta}_2)=R(\mathbf{\Theta}_1)R(\mathbf{\Theta}_2)R^{-1}(\mathbf{\Theta}_1)=\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\mathbf{\Theta}_1/2\right)\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\mathbf{\Theta}_2/2\right)\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\mathbf{\Theta}_1/2\right) \tag{3-57}$$

Two successive rotations yields:

$$R'(\mathbf{\Theta}_2)R(\mathbf{\Theta}_1)=\left[R(\mathbf{\Theta}_1)R(\mathbf{\Theta}_2)R^{-1}(\mathbf{\Theta}_1)\right]R(\mathbf{\Theta}_1)=\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\mathbf{\Theta}_1/2\right)\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\mathbf{\Theta}_2/2\right) \tag{3-58}$$

Axes which are rotated along with the spinors are called embedded axes. Rotation angles which refer to embedded axes are called *Euler angles*. We use primes to denote rotations about embedded axes.

The Euler rotation operator $R'(\mathbf{\Theta}_2)$ can be interpreted as follows: First, rotate the spinor back to its original orientation. Next, rotate the spinor about the fixed axis corresponding to $\mathbf{\Theta}_2$. Finally, rotate again about the embedded axis corresponding to $\mathbf{\Theta}_1$ (the original axis now rotated by $\mathbf{\Theta}_2$). The equation states that rotation by $\mathbf{\Theta}_1=\mathbf{\Theta}_1'$ followed by rotation about the fixed axis $\mathbf{\Theta}_2$ is equivalent to rotation first by $\mathbf{\Theta}_2$ followed by rotation by $\mathbf{\Theta}_1'$ about the embedded $\hat{\mathbf{\Theta}}_1'$ axis. In the above example, rotation by $\pi/2$ about *x* followed by $\pi/2$ about *z* (or y') is equivalent to rotation by $\pi/2$ about *z* followed by $\pi/2$ about *y* (or x').

The angular derivative of the wave function is:

$$\partial_{\boldsymbol{\varphi}}\psi=\partial_{\boldsymbol{\varphi}}\left[\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\psi_{\mathrm{v}}\right]=-\tilde{\mathrm{i}}\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\frac{\boldsymbol{\beta}}{2}\psi_{\mathrm{v}}=-\tilde{\mathrm{i}}\left[\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\frac{\boldsymbol{\beta}}{2}\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\right]\psi \tag{3-59}$$

It is customary in quantum mechanics to define the angular derivative to be:

$$\partial_{\boldsymbol{\varphi}'}\psi=-\tilde{\mathrm{i}}\,\frac{\boldsymbol{\beta}}{2}\psi \tag{3-60}$$

This relation is only valid if the angle $\boldsymbol{\varphi}'$ is measured with respect to the embedded axes.

Accumulated rotations can be computed from successive rotations about embedded axes. Given a rotation rate $\mathbf{w}'(t)$ with respect to embedded axes, the accumulated rotation operator is:

$$R(t) = R(\mathbf{\Theta}'(t)) = \exp(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\mathbf{\Theta}'/2) = \exp(\tilde{\mathrm{i}}\int dt\,\boldsymbol{\beta}\cdot\mathbf{w}'/2) \tag{3-61}$$

3.4.4. Examples

Let us verify this expression with explicit examples. First, we compute the general expression for rotation about two successive embedded axes: Rotate by angle $\boldsymbol{\theta}'_a$ about an axis x'_a followed by $\boldsymbol{\theta}'_b$ about x'_b. The rotation operator is:

$$\begin{aligned} R(t) = R(\mathbf{\Theta}'(t)) &= \exp(\tilde{\mathrm{i}}\,\beta_b\theta'_b/2)\exp(\tilde{\mathrm{i}}\,\beta_a\theta'_a/2) \\ &= \left[\cos\frac{\theta'_b}{2} + \tilde{\mathrm{i}}\,\beta_b\sin\frac{\theta'_b}{2}\right]\left[\cos\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\,\beta_a\sin\frac{\theta'_a}{2}\right] \\ &= \cos\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2} - \beta_b\beta_a\sin\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\left[\beta_a\cos\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \beta_b\sin\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2}\right] \end{aligned} \tag{3-62}$$

Recall that $\beta_a\beta_b = \beta_a\cdot\beta_b + \tilde{\mathrm{i}}\,\beta_a\times\beta_b$. We consider two special cases. First, if $\boldsymbol{\theta}'_a$ and $\boldsymbol{\theta}'_b$ are parallel then:

$$\begin{aligned} R(\mathbf{\Theta}'(t)) &= \cos\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2} - \sin\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \mathrm{i}\,\beta_a\left[\cos\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \sin\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2}\right] \\ &= \cos\left(\frac{\theta'_b + \theta'_a}{2}\right) + \mathrm{i}\,\beta_a\sin\left(\frac{\theta'_b + \theta'_a}{2}\right) \end{aligned} \tag{3-63}$$

which is obviously correct since parallel angles are additive. Next consider two perpendicular axes with $\beta_b\beta_a = \tilde{\mathrm{i}}\,\beta_c$:

$$R(\mathbf{\Theta}'(t)) = \cos\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\,\beta_c\sin\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\left[\beta_a\cos\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \beta_b\sin\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2}\right] \tag{3-64}$$

For the special case where both angles are $\pi/2$ this yields:

$$\begin{aligned} R(\mathbf{\Theta}'(t)) &= \cos^2\frac{\pi}{4} + \tilde{\mathrm{i}}\,\beta_c\sin^2\frac{\pi}{4} + \tilde{\mathrm{i}}\left[\beta_a\cos\frac{\pi}{4}\sin\frac{\pi}{4} + \beta_b\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right] \\ &= \frac{1}{2} + \tilde{\mathrm{i}}\,\frac{\beta_a + \beta_b + \beta_c}{2} = \cos(\pi/3) + \tilde{\mathrm{i}}\,\frac{\beta_a + \beta_b + \beta_c}{\sqrt{3}}\sin(\pi/3) \end{aligned} \tag{3-65}$$

This corresponds to a rotation operator for $2\pi/3$ radians about the axis $[\hat{\mathbf{x}}_a + \hat{\mathbf{x}}_b + \hat{\mathbf{x}}_c]/\sqrt{3}$. The validity of this result can be verified by picturing an equilateral triangle with corners on each axis equidistant from the origin. Clearly rotation by $2\pi/3$ about the center of the triangle merely permutes the positions of the axes, which is of course what happens when rotating by $\pi/2$ around successive orthogonal axes. Note also that the symmetry of the final result implies that:

$$\exp\left(\tilde{\mathrm{i}}\,\beta_y\,\pi/4\right)\exp\left(\tilde{\mathrm{i}}\,\beta_x\,\pi/4\right)=\exp\left(\tilde{\mathrm{i}}\,\beta_z\,\pi/4\right)\exp\left(\tilde{\mathrm{i}}\,\beta_y\,\pi/4\right)$$
$$=\exp\left(\tilde{\mathrm{i}}\,\beta_x\,\pi/4\right)\exp\left(\tilde{\mathrm{i}}\,\beta_z\,\pi/4\right) \qquad (3\text{-}66)$$

(x followed by y', y followed by z', z followed by x') which is consistent with our explanation of the secondary rotation operator above.

3.4.5. *Wave Function*

In three dimensions the gradient can be defined as a one-dimensional derivative rotated by angle $\boldsymbol{\zeta}$ to a new axis $\hat{\mathbf{v}}$. Let:

$$\beta_{\mathrm{v}} \equiv \exp\left(-\,\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\beta_3\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)$$
$$\psi \equiv \exp\left(-\,\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_{\mathrm{v}} \qquad (3\text{-}67)$$

Rotation by angle $\boldsymbol{\zeta}$ is denoted R_{ζ} and defined relative to a default orientation along the x_3 axis. The three-dimensional gradient is:

$$\nabla a = R_{\zeta}\left\{\hat{\mathbf{x}}_3\frac{\partial a}{\partial x_3}\right\} = -\hat{\mathbf{v}}\psi_{\mathrm{v}}^T c\beta_3\psi_{\mathrm{v}} = -c\psi^{\dagger}\boldsymbol{\beta}\psi \qquad (3\text{-}68)$$

Writing a column matrix as the transpose of a row matrix, the rotated wave function ψ is:

$$\psi = \dot{a}_0^{1/2}\exp\left(-\,\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\exp\left(\beta_3\sigma\alpha/2\right)\begin{bmatrix}\delta_{\mathrm{F}} & \delta_B & \left[1-\delta_{\mathrm{F}}\right] & \left[1-\delta_B\right]\end{bmatrix}^T/\sqrt{2} \qquad (3\text{-}69)$$

However, in three dimensions the constant column matrix which represents $v_3 = 0$ states may have nonzero velocity perpendicular to x_3. This is indeed the case for $\psi_0 = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^T/\sqrt{2}$ and $\psi_0 = \begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}^T/\sqrt{2}$. The remaining states with zero velocity are obtained by rotation of velocity from:

$$\psi_0 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T/\sqrt{2} \qquad (3\text{-}70)$$

This state has zero time derivative but nonzero gradient. When Lorentz boosts are applied both the time derivative and velocity can be non-zero. The final form of the wave function is thus:

$$\psi = \dot{a}_0^{1/2}\exp\left(-\,\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\exp\left(\beta_3\sigma\alpha/2\right)\psi_0 \qquad (3\text{-}71)$$

This is the general form of the scalar wave function. The constant matrix is multiplied by factors representing an amplitude, a 1-D velocity boost, and a general rotation in velocity space (two angles to determine velocity direction plus rotation about the velocity axis). Clearly four parameters are needed to determine $\partial_t a$ and ∇a. The significance of rotation about the velocity axis will be discussed below.

3.4.6. *First-Order Wave Equation*

The time derivative of (3-67) yields the first-order equation:

$$\partial_t\psi = -\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\partial_t\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi_{\mathrm{v}} + \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\partial_t\psi_{\mathrm{v}} \tag{3-72}$$

Here we can see the effect of rotation about the velocity axis. Rotation of the left-hand side involves only direct rotation of the wave function, but rotation of the right-hand side also involves rotation of the angular frequency $\partial_t\boldsymbol{\zeta}$. Rotation about the velocity (or gradient) axis can change the direction of this angular frequency. This is the significance of the fifth parameter in the factorization above.

Inverting the rotation factor yields the one-dimensional wave function, which satisfies the one-dimensional wave equation:

$$\left[\partial_t + c\beta_3\sigma\hat{\mathbf{v}}\cdot\nabla\right]\left[\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi\right] = 0 \tag{3-73}$$

Derivatives of the exponential factors are:

$$\begin{aligned}\partial_t\exp\left(\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right) &= \frac{\tilde{\mathrm{i}}}{2}\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\partial_t\left[\boldsymbol{\beta}\cdot\boldsymbol{\zeta}\right]\\ \nabla\exp\left(\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right) &= \frac{\tilde{\mathrm{i}}}{2}\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\nabla\left[\boldsymbol{\beta}\cdot\boldsymbol{\zeta}\right]\end{aligned} \tag{3-74}$$

Substituting $\sigma\beta_3\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\hat{\mathbf{v}}\cdot\nabla = \exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\sigma\boldsymbol{\beta}\cdot\nabla$ into (3-73) yields:

$$\partial_t\psi + c\sigma\boldsymbol{\beta}\cdot\nabla\psi = -\partial_t\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\sigma\boldsymbol{\beta}\cdot\nabla\left[\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\right]\psi \tag{3-75}$$

This equation states that the convective derivative is nonzero only due to (convective) rotation of velocity direction.

The equation of evolution of the scalar wave amplitude is obtained by multiplying $\psi^\dagger\sigma$ and adding the adjoint:

$$\begin{aligned}\partial_t\left[\psi^\dagger\sigma\psi\right] &= \left\{-c\sigma\boldsymbol{\beta}\cdot\nabla\psi - \frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\sigma\boldsymbol{\beta}\cdot\nabla\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi\right\}^\dagger\sigma\psi\\ &\quad+\psi^\dagger\sigma\left\{-c\sigma\boldsymbol{\beta}\cdot\nabla\psi - \frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\sigma\boldsymbol{\beta}\cdot\nabla\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi\right\}\\ &= -\nabla\cdot\psi^\dagger c\boldsymbol{\beta}\psi + c\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\left[\psi^\dagger\boldsymbol{\beta}\psi\right]\end{aligned} \tag{3-76}$$

In terms of the scalar polarization, this equation is:

$$\partial_t^2 a = c^2\nabla^2 a - c^2\left[\nabla\times\boldsymbol{\zeta}\right]\cdot\nabla a \tag{3-77}$$

The relations between rotation angles and velocity unit vectors are:

$$\partial_t \boldsymbol{\zeta} = \hat{e}_{\mathrm{v}} \times \partial_t \hat{e}_{\mathrm{v}}$$
$$\nabla \cdot \boldsymbol{\zeta} = -\hat{e}_{\mathrm{v}} \cdot [\nabla \times \hat{e}_{\mathrm{v}}]$$
$$\nabla \times \boldsymbol{\zeta} = \hat{e}_{\mathrm{v}}[\nabla \cdot \hat{e}_{\mathrm{v}}] - [\hat{e}_{\mathrm{v}} \cdot \nabla]\hat{e}_{\mathrm{v}}$$
$$[\nabla \times \boldsymbol{\zeta}] \cdot \nabla = \nabla^2 - [\hat{e}_{\mathrm{v}} \cdot \nabla][\hat{e}_{\mathrm{v}} \cdot \nabla] = [\hat{e}_{\mathrm{v}} \times \nabla] \cdot [\hat{e}_{\mathrm{v}} \times \nabla] \tag{3-78}$$

So that the above equation is indeed equivalent to the one-dimensional wave equation:

$$\partial_t^2 a = c^2 [\hat{e}_{\mathrm{v}} \cdot \nabla][\hat{e}_{\mathrm{v}} \cdot \nabla] a \tag{3-79}$$

If we want to obtain the conventional 3D scalar wave equation:

$$\partial_t^2 a = c^2 \nabla^2 a \tag{3-80}$$

Then the simplest corresponding first order equation is:

$$\partial_t \psi + c\sigma \boldsymbol{\beta} \cdot \nabla \psi = 0 \tag{3-81}$$

3.5. Vector Waves

"*Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the "Old One." I, at any rate, am convinced that He is not playing at dice. Waves in three-dimensional space whose velocity is regulated by potential energy (for example, rubber bands) . . .*"
– Albert Einstein, 1926 [Einstein and Born 2005]

Next we consider vector waves (polar or axial vectors). An arbitrary polarization vector can be described by a scalar amplitude and three rotation angles. Since scalar waves require five parameters, we expect vector waves to require eight parameters. As with velocity rotations, only two angles are necessary to determine the direction of polarization, but a third angle is necessary for a local description of changes in the polarization direction.

3.5.1. Rotation of Polarization

Recall that the scalar polarization is $\dot{a} = \psi^T \sigma \psi$. We now regard this as one component of a vector: $\dot{a}_3 = \psi^T \sigma_3 \psi$. The vector **a** could be polar or axial, but we will assume an axial vector (pseudovector). The three orthogonal polarization matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\bar{\mathrm{i}} & 0 & 0 \\ \bar{\mathrm{i}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\mathrm{i}} \\ 0 & 0 & \bar{\mathrm{i}} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3-82}$$

The symbol ($\bar{\mathrm{i}}$) is a unit scalar imaginary which is even under spatial inversion since the spin is a pseudovector.

These matrices have the same commutation relations as the Pauli matrices:

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \; ; \; \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\sigma_k \tag{3-83}$$

The rotation operators for this space are similar to the velocity matrix rotation operators:

$$R_{\sigma_j}(\xi_i) = \exp\left(-\bar{\mathrm{i}}\,\sigma_i\xi_i/2\right)\sigma_j \exp\left(\bar{\mathrm{i}}\,\sigma_i\xi_i/2\right) = \sigma_j \cos\xi_i - \frac{\bar{\mathrm{i}}}{2}\left[\sigma_i\sigma_j - \sigma_j\sigma_i\right]\sin\xi_i \tag{3-84}$$

We could simply generalize the wave function to be:

$$\psi = \dot{a}_0^{1/2}\exp\left(-\bar{\mathrm{i}}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\exp\left(\beta_3\sigma\alpha/2\right)\psi_0 \tag{3-85}$$

We might then attempt the interpretation:

$$\partial_i S_j = -\psi^\dagger\beta_i\sigma_j\psi \tag{3-86}$$

However, there are nine tensor components (plus three components of the time derivative) and only eight independent components of the bispinor. Therefore this interpretation is not satisfactory unless additional constraints are imposed.

Instead, we will assume a single rotation operator for both wave velocity and polarization. Since the one-dimensional velocity is $v = c\beta_3\sigma_3$, the three-dimensional velocity for vector waves is $\mathbf{v} = c\beta_3\boldsymbol{\sigma}$. The $\boldsymbol{\beta}$ matrices which described velocity for scalar waves now represent directions relative to velocity, with β_3 representing the parallel direction. This notation is called the "chiral representation" of velocity.

Alternatively, we could associate any of the matrices β_i with velocity by rotating in the relative-velocity space of $\boldsymbol{\beta}$ matrices. Such a rotation is called a change of "representation". The form $\mathbf{v} = c\beta_1\boldsymbol{\sigma}$ has the form originally used by Dirac, and we will use these matrices for velocity. Historically, a different notation has been used for the $\boldsymbol{\beta}$ matrices. Instead of $(\beta_1, \beta_2, \beta_3)$, these matrices have been called $\left(\gamma^5, \mathrm{i}\gamma^5\gamma^0, \gamma^0\right)$. However, we will continue to call them $\boldsymbol{\beta}$ matrices except when comparing with standard results from other literature.

3.5.2. Factorization and First-Order Wave Equation

The three-dimensional bispinor wave function may have a Lorentz boost with arbitrary magnitude and direction, and may also be rotated by an arbitrary angle $\boldsymbol{\xi}$. These operators are contained in the factorization:

$$\psi = \dot{a}_0^{1/2}\exp\left(-\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(\beta_1\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2\right)\psi_0 \tag{3-87}$$

The wave function has seven free parameters: an amplitude, three rotation angles, and three velocity parameters. There is one additional degree of freedom which determines the definition of the relative directions β_2 and β_3. These are defined with respect to the velocity axis by the operator $\exp\left(\tilde{\mathrm{i}}\,\beta_1\,\zeta/2\right)\psi_0$, so that the wave function is [Hestenes 1967]:

$$\psi = \dot{a}_0^{1/2}\exp\left(-\bar{\mathrm{i}}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(\beta_1\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2\right)\exp\left(\tilde{\mathrm{i}}\,\beta_1\,\zeta/2\right)\psi_0 \tag{3-88}$$

Now we would like to know the equation of evolution of the wave function. Generalizing the scalar wave equation (3-73) to include arbitrary gradient direction yields:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi \tag{3-89}$$

These terms account for wave propagation in an arbitrary direction.

To see the wave equation in terms of observables, multiply $\psi^\dagger \sigma_j$ and add the transpose equation to obtain the time derivative of the polarization:

$$\partial_t \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = -c\nabla\left[\psi^\dagger \beta_1 \psi\right] - \bar{\mathrm{i}}\, c\left[\nabla \psi^\dagger \times \beta_1 \boldsymbol{\sigma} \psi + \psi^\dagger \beta_1 \boldsymbol{\sigma} \times \nabla \psi\right] \tag{3-90}$$

The terms in this equation are naturally associated with spinors by the following definitions:

$$\begin{aligned} \partial_t^2 a_j &\equiv \partial_t \left[\psi^\dagger \sigma_j \psi\right] \\ c^2 \partial_j [\nabla \cdot \mathbf{a}] &\equiv -c\partial_j \left[\psi^\dagger \beta_1 \psi\right] \\ c^2 \{\nabla \times \nabla \times \mathbf{a}\}_j &\equiv -\bar{\mathrm{i}}\, c\varepsilon_{ijk} \left\{\partial_i \psi^\dagger \beta_1 \sigma_k \psi - \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right\} \end{aligned} \tag{3-91}$$

These identifications yield the wave equation:

$$\partial_t^2 \mathbf{a} = c^2 \nabla^2 \mathbf{a} \tag{3-92}$$

$$\partial_t \left[\psi^\dagger \beta_1 \psi\right] + c\nabla \cdot \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = 0 \tag{3-93}$$

This relation is easily derived from equation (3-89).

Also from (3-89):

$$\partial_t \left[\psi^\dagger \psi\right] + c^2 \nabla \cdot \left[\psi^\dagger \beta_1 \boldsymbol{\sigma} \psi\right] = 0 \tag{3-94}$$

This is the quantum mechanical continuity equation. This is the three dimensional generalization of the 1-D equation:

$$\partial_t |\partial_t a_F| + \partial_t |\partial_t a_B| + c^2 \partial_z |\partial_z a_F| - c^2 \partial_z |\partial_z a_B| = 0 \tag{3-95}$$

3.5.3. Convection and Rotation

Adding terms for convection and rotation to the bispinor wave equation yields:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi - \mathbf{u} \cdot \nabla \psi - \mathbf{w} \cdot \partial_\varphi \psi \tag{3-96}$$

From the wave factorization we can substitute minus the angular derivative (for passive rotation) in the final term:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi - \mathbf{u} \cdot \nabla \psi - \bar{\mathrm{i}}\, \mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \psi \tag{3-97}$$

To see the wave equation in terms of observables, multiply $\psi^\dagger\sigma_j$ and add the transpose equation to obtain the time derivative of the polarization:

$$\partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right]=-c\nabla\left[\psi^\dagger\beta_1\psi\right]+\bar{\mathrm{i}}\,c\varepsilon_{ijk}\left[\partial_i\psi^\dagger\beta_1\sigma_k\psi-\psi^\dagger\beta_1\sigma_k\partial_i\psi\right]\,\hat{\mathbf{e}}_j \\ -\mathbf{u}\cdot\nabla\left[\psi^\dagger\boldsymbol{\sigma}\psi\right]+\mathbf{w}\times\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] \tag{3-98}$$

These identifications yield the equation of a wave propagating in a moving medium:

$$\partial_t^2\mathbf{a}=c^2\nabla^2\mathbf{a}-\mathbf{u}\cdot\nabla\dot{\mathbf{a}}+\mathbf{w}\times\dot{\mathbf{a}} \tag{3-99}$$

Using equation (3-97) now yields different continuity conditions:

$$\partial_t\left[\psi^\dagger\beta_1\psi\right]+c^2\nabla\cdot\left[\psi^\dagger\boldsymbol{\sigma}\psi\right]+\mathbf{u}\cdot\nabla\left[\psi^\dagger\beta_1\psi\right]=0 \tag{3-100}$$

Consistency with our definition of variables requires that:

$$\mathbf{u}\cdot\nabla\left[\psi^\dagger\beta_1\psi\right]=0 \tag{3-101}$$

Also from (3-97):

$$\partial_t\left[\psi^\dagger\psi\right]+c^2\nabla\cdot\left[\psi^\dagger\beta_1\boldsymbol{\sigma}\psi\right]+\mathbf{u}\cdot\nabla\left[\psi^\dagger\psi\right]=0 \tag{3-102}$$

The continuity equation now includes an additional convection term.

Next, we will interpret the wave polarization.

3.6. Waves in an Elastic Solid

"I am never content until I have constructed a mechanical model of the subject I am studying. If I succeed in making one, I understand; otherwise I do not."
– William Thomson (Lord Kelvin) 1904

In this section we will analyze rotational, or torsion, waves in an ideal elastic solid. This section is based on previously published work by the author. The basic ideas were published in Close [2008] and the full Lagrangian and dynamical operators were published in Close [2011a].

3.6.1. Basic Assumptions

We make the following basic assumptions:

1. The elastic solid is characterized by an inertial density ρ and coefficient of elasticity μ, with characteristic wave speed $c=\sqrt{\mu/\rho}$.
2. There is a linear response to variations of orientation angle $\boldsymbol{\Theta}$ relative to equilibrium. This means that an initial static perturbation (with velocity **u**=0) would yield the response:

$$\partial_t^2\boldsymbol{\Theta}=c^2\nabla^2\boldsymbol{\Theta}\quad\left(\text{if }\mathbf{u}=0\right) \tag{3-103}$$

3. The velocity field **u** has no compression: $\nabla \cdot \mathbf{u} = 0$. Therefore the velocity may be written as the curl of a vector field:

$$\mathbf{u} = \frac{1}{2\rho}[\nabla \times \mathbf{S}] \tag{3-104}$$

The vector field **S** is called the spin angular momentum density. It differs from the conventional definition of angular momentum density $\mathbf{r} \times \rho\mathbf{u}$ in that it is independent of the choice of origin and can have arbitrary direction at any point. If $|\mathbf{u}|$ falls to zero sufficiently rapidly toward infinity, then kinetic energy may be expressed as:

$$K = \int d^3r\, \rho u^2/2 \quad = \frac{1}{2}\int d^3r\, \mathbf{w} \cdot \mathbf{S}$$

where $\mathbf{w} = \nabla \times \mathbf{u}/2$ is the angular velocity, or vorticity (see Chapter 1).

Additional assumptions will be introduced in order to simplify the mathematics, and these may limit the generality of the results.

3.6.2. *Equation of Evolution*

Starting from (3-165), we define an angular potential **Q** such that:

$$\nabla^2 \mathbf{Q} = -4\rho\boldsymbol{\Theta} \tag{3-105}$$

The static condition for **Q** is:

$$\nabla^2 \left\{ \partial_t^2 \mathbf{Q} - c^2 \nabla^2 \mathbf{Q} \right\} = 0 \quad (\text{if } \mathbf{u} = 0) \tag{3-106}$$

Define the spin angular momentum as:

$$\mathbf{S} \equiv \partial_t \mathbf{Q} \tag{3-107}$$

The static condition is then:

$$\nabla^2 \left\{ \partial_t \mathbf{S} - c^2 \nabla^2 \mathbf{Q} \right\} = 0 \quad (\text{if } \mathbf{u} = 0) \tag{3-108}$$

When motion is present, it contributes to the time derivative only through convection ($-\mathbf{u} \cdot \nabla\mathbf{S}$) and rotation ($\mathbf{w} \times \mathbf{S}$). This assumes that there are no velocity-dependent forces such as frictional damping. From here on, we will consider only wave-like solutions satisfying:

$$\partial_t \mathbf{S} - c^2 \nabla^2 \mathbf{Q} + \mathbf{u} \cdot \nabla\mathbf{S} - \mathbf{w} \times \mathbf{S} = 0 \tag{3-109}$$

For oscillatory solutions to this equation, the first two terms are always in phase $\left(\partial_t^2 \mathbf{Q} - c^2 \nabla^2 \mathbf{Q}\right)$, whereas the nonlinear term may have different phase. However, if the nonlinear term is not zero then it must have the same phase as the linear terms:

$$\mathbf{u} \cdot \nabla - \mathbf{w} \times \mathbf{S} = \Omega^2(\mathbf{r})\mathbf{Q} \tag{3-110}$$

where $\Omega^2(\mathbf{r})$ is some function of position (more generally, $\Omega^2(\mathbf{r})$ could have different values for each component of **Q**). Substitution yields:

$$\partial_t^2 \mathbf{Q} - c^2 \nabla^2 \mathbf{Q} + \Omega^2(\mathbf{r})\mathbf{Q} = 0 \tag{3-111}$$

If $\Omega^2(\mathbf{r})$ is constant and positive, then this is the Klein-Gordon equation, which is ordinarily associated with bosons.

The wave equation (3-109) can be written in terms of a four-component complex Dirac bispinor (ψ) using the following identifications:

$$\begin{aligned} \partial_t^2 Q_j &\equiv \frac{1}{2}\partial_t\left[\psi^\dagger \sigma_j \psi\right] \\ c^2 \partial_j [\nabla \cdot \mathbf{Q}] &\equiv -\frac{1}{2} c \partial_j \left[\psi^\dagger \beta_1 \psi\right] \\ c^2 \{\nabla \times \nabla \times \mathbf{Q}\}_j &\equiv -\frac{\mathrm{i}}{2} c \varepsilon_{ijk} \left\{\partial_i \psi^\dagger \beta_1 \sigma_k \psi - \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right\} \end{aligned} \tag{3-112}$$

The matrices $c\beta_1\sigma_j$ are the Dirac velocity matrices, more conventionally denoted as $c\gamma^5\sigma_j$.

The above identifications provide 7 constraints on the 8 free parameters of the Dirac bispinor. In terms of bispinors, the rotational wave equation (3-**Error! Bookmark not defined.**) is:

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left[\psi^\dagger \sigma_j \psi\right] + \frac{1}{2} c \partial_j \left[\psi^\dagger \beta_1 \psi\right] - \frac{1}{2} \mathrm{i} c \varepsilon_{ijk} \left\{\partial_i \psi^\dagger \beta_1 \sigma_k \psi + \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right\} \\ &+ \frac{1}{2} \mathbf{u} \cdot \nabla \left[\psi^\dagger \sigma_j \psi\right] - \frac{1}{2} \varepsilon_{kij} w_k \left[\psi^\dagger \sigma_i \psi\right] = 0 \end{aligned} \tag{3-113}$$

Expanding the derivatives yields:

$$\frac{1}{2}\psi^\dagger \sigma_j \left[\partial_t \psi + c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi + \mathbf{u} \cdot \nabla \psi + \mathbf{w} \cdot \frac{\mathrm{i}\boldsymbol{\sigma}}{2} \psi\right] + \mathrm{h.c.} = 0 \tag{3-114}$$

where (h.c.) represents the Hermitian conjugate. The Hermitian conjugate wave function may be regarded as an independent variable (the independent real and imaginary parts of the wave function are linear combinations of elements of ψ and $\psi^\dagger$). Validity for arbitrary $\psi^\dagger$ requires the terms in brackets to sum to zero. This yields the Dirac equation:

$$\partial_t \psi + c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi + \mathbf{u} \cdot \nabla \psi + \mathbf{w} \cdot \frac{\mathrm{i}\boldsymbol{\sigma}}{2} \psi + \mathrm{i}\chi\psi = 0 \tag{3-115}$$

where χ may be any operator with the property:

$$\mathrm{Re}\left(\psi^\dagger \sigma_j \mathrm{i}\chi\psi\right) = 0 \tag{3-116}$$

Since χ does not contribute to angular momentum density, we assume it to be zero.

For convencience, we multiply the Dirac equation by the unit imaginary times the adjount wave function:

$$\psi^\dagger \mathrm{i}\partial_t\psi + \psi^\dagger \mathrm{i}c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi + \psi^\dagger\mathbf{u}\cdot\mathrm{i}\nabla\psi - \psi^\dagger\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\psi = 0 \tag{3-117}$$

Now we construct a Lagrange density. Lagrange's equations of motion for a field variable ψ are:

$$\partial_t\frac{\partial\mathscr{L}}{\partial\left[\partial_t\psi^\dagger\right]} + \sum_j \partial_j\frac{\partial\mathscr{L}}{\partial\left[\partial_j\psi^\dagger\right]} - \frac{\partial\mathscr{L}}{\partial\psi^\dagger} = 0 \tag{3-118}$$

A similar equation holds with ψ replacing $\psi^\dagger$. It is possible to construct a Lagrangian with no derivatives of $\psi^\dagger$, so that the equation of motion is simply $\partial\mathscr{L}/\partial\psi^\dagger = 0$. The nonlinear terms (with u and w) contain two factors of $\psi^\dagger$. In the rotation term, these may be interchanged using integration by parts. Therefore this term requires a factor of one-half in the Lagrangian. In the convection term, however, integration by parts yields a term containing $\nabla\cdot\mathbf{u}$. Since this is zero, the factor of $\psi^\dagger$ contained in it does not contribute to the Euler-Lagrange equation.

Hence we obtain:

$$\mathscr{L} = \psi^\dagger \mathrm{i}\partial_t\psi + \psi^\dagger \mathrm{i}c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi + \psi^\dagger\mathbf{u}\cdot\mathrm{i}\nabla\psi - \frac{1}{2}\psi^\dagger\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\psi \tag{3-119}$$

This Lagrangian is not real, but we may take the real part as representing physical quantities. Notice that the final term represents minus the kinetic energy, so this is not the classical form (*K*-*U*). This will require some care with the signs of conjugate momenta.

The conjugate momentum to the field $\psi^\dagger$ is p_ψ:

$$p_\psi = \frac{\partial\mathscr{L}}{\partial\left[\partial_t\psi\right]} = \mathrm{i}\psi^\dagger \tag{3-120}$$

We recognize the last term in the Lagrangian as the kinetic energy density *K*. A consistent (when integrated) interpretation of the other terms is:

$$\begin{aligned} &\mathrm{Re}\left\{\psi^\dagger \mathrm{i}\partial_t\psi + \psi^\dagger \mathrm{i}c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi + \psi^\dagger\mathbf{u}\cdot\mathrm{i}\nabla\psi\right\} - \frac{1}{2}\psi^\dagger\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\psi = 0 \\ &\qquad \mathrm{E} \quad - \quad \mathbf{v}\cdot\mathbf{p} \quad - \quad \mathbf{u}\cdot\mathbf{p} \quad - \quad \frac{1}{2}\mathbf{u}\cdot\mathbf{q} \quad = 0 \end{aligned} \tag{3-121}$$

where **u** and **q** are the velocity and momentum of the medium, respectively, while **v** and **p** are the velocity and momentum of the wave, respectively. There is no $\mathbf{v}\cdot\mathbf{q}$ term, presumably because the wave is transverse (wave velocity orthogonal to medium velocity). The wave energy and momentum density are taken to be:

$$\begin{aligned} E &= \mathrm{Re}\left\{\psi^\dagger \mathrm{i}\partial_t\psi\right\} \\ \mathbf{p} &= -\mathrm{Re}\left\{\psi^\dagger \mathrm{i}\nabla\psi\right\} \end{aligned} \tag{3-122}$$

These will be derived below. Since the $\mathbf{v}\cdot\mathbf{p}$ term involves only spatial derivatives of the wave function, it is more appropriate to interpret it as elastic potential energy density U:

$$E - U - \mathbf{u}\cdot\mathbf{p} - K = 0 \tag{3-123}$$

The convection term transports momentum and energy, but we hypothesize that it integrates to zero, thereby having no effect on the total energy.

3.6.1. *Dynamical Variables*

Energy and momentum

The Hamiltonian is:

$$\mathscr{H} = p_\psi \partial_t \psi - \mathscr{L} = -\psi^\dagger \,\mathrm{i} c\beta_1 \boldsymbol{\sigma}\cdot\nabla\psi - \psi^\dagger \mathbf{u}\cdot \mathrm{i}\nabla\psi + \frac{1}{2}\psi^\dagger \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\psi = U + \mathbf{u}\cdot\mathbf{p} + K \tag{3-124}$$

Hamilton's equation for the wave function is:

$$\partial_t \psi = \frac{\partial \mathscr{H}}{\partial p_\psi} = \frac{\partial \mathscr{H}}{\partial\left[\mathrm{i}\psi^\dagger\right]} = \left\{-c\beta_1\boldsymbol{\sigma}\cdot\nabla - \mathbf{u}\cdot\nabla - \mathrm{i}\,\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\right\}\psi \tag{3-125}$$

We can also define a Hamiltonian operator with $\mathrm{i}\partial_t\psi = H\psi$ (as in quantum mechanics):

$$H = -\mathrm{i} c\beta_1\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\mathbf{u}\cdot\nabla + \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2} \tag{3-126}$$

The Hamiltonian is a special case (T_0^0) of the energy-momentum tensor:

$$T_\nu^\mu = \frac{\partial\mathscr{L}}{\partial\left[\partial_\mu\psi\right]}\partial_\nu\psi - \mathscr{L}\delta_\nu^\mu \tag{3-127}$$

The dynamical momentum density is:

$$p_i = -T_i^0 = -\frac{\partial\mathscr{L}}{\partial\left[\partial_t\psi\right]}\partial_i\psi = -\mathrm{i}\,\psi^\dagger\partial_i\psi \tag{3-128}$$

This is identical to the momentum density of relativistic quantum mechanics. The dynamical angular momentum is similarly:

$$\mathbf{L} = -\frac{\partial\mathscr{L}}{\partial\left[\partial_t\psi\right]}\partial_\varphi\psi(\mathbf{r},t) = -\mathrm{i}\,\psi^\dagger\frac{\partial r_i}{\partial\boldsymbol{\varphi}}\frac{\partial}{\partial r_i}\psi(\mathbf{r},t) = -\mathrm{i}\,\psi^\dagger\left[\mathbf{r}\times\nabla\right]\psi \tag{3-129}$$

For the total momentum and angular momentum, we must add the contribution of the motion of the medium:

$$\mathbf{P} = \mathbf{p} + \mathbf{q} = -\psi^\dagger \,\mathrm{i}\nabla\psi + \frac{1}{2}\nabla\times\psi^\dagger \frac{\boldsymbol{\sigma}}{2}\psi$$
$$\mathbf{J} = \mathbf{L} + \mathbf{S} = -\mathbf{r}\times\psi^\dagger \,\mathrm{i}\nabla\psi + \psi^\dagger \frac{\boldsymbol{\sigma}}{2}\psi \tag{3-130}$$

Active rotations are described by the operator $U(\boldsymbol{\varphi})$ [Schiff]:

$$\partial_\varphi U_\varphi \psi(\mathbf{r},t) = -\mathbf{r}\times\nabla\psi - \mathrm{i}\frac{\boldsymbol{\sigma}}{2}\psi = -\mathrm{i}(L+S)\psi$$
$$U_\varphi \psi(\mathbf{r},t) = \exp(-\mathrm{i}(\mathbf{L}+\mathbf{S})\cdot\boldsymbol{\varphi})\psi(\mathbf{r},t) \tag{3-131}$$

In summary, we have demonstrated that rotational waves in an elastic solid are described by a Lagrangian with exactly the same dynamical operators (within a normalization factor) as found in relativistic quantum mechanics.

3.7. Electron Waves

"... a great step would be made when we should be able to say of electricity that which we say of light, in saying that it consists of undulations."
—Sir George Gabriel Stokes, 1879

3.7.1. Free Electron Equation

The bispinor equation for angular momentum density is:

$$\partial_t\psi = -c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi - \mathbf{u}\cdot\nabla\psi - \mathrm{i}\,\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\psi \tag{3-132}$$

A formal solution is:

$$\psi(\mathbf{r},t) = \exp\left\{\int_{t_0}^{t} dt\left(-c\beta_1\boldsymbol{\sigma}\cdot\nabla - \mathbf{u}\cdot\nabla\psi - \mathrm{i}\,\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right\}\psi(\mathbf{r},t_0) \tag{3-133}$$

3.7.2. Mass, Convection, and Rotation

Dirac's derivation of the mass term simply required that each component of the wave function satisfy the Klein-Gordon equation. One possible formulation would be:

$$\partial_t\psi = -c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi - \mathrm{i}\,\mu\beta_1\boldsymbol{\sigma}\cdot[\hat{\mathbf{e}}_3\times\nabla]\psi \tag{3-134}$$

The second-order equation is:

$$\partial_t^2\psi = \left[c^2\nabla^2\psi - \mu^2[\hat{\mathbf{e}}_3\times\nabla]^2\right]\psi \tag{3-135}$$

which is equivalent to Klein-Gordon if the wave function is an eigenfunction of the operator $[\hat{\mathbf{e}}_3\times\nabla]^2$. The equivalent classical equation is:

$$\partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = -c\nabla\left[\psi^\dagger\beta_1\psi\right] + \bar{\mathrm{i}}\,c\varepsilon_{ijk}\left[\partial_i\psi^\dagger\beta_1\sigma_k\psi - \psi^\dagger\beta_1\sigma_k\partial_i\psi\right]\hat{\mathbf{e}}_j + \mu\left[\hat{\mathbf{e}}_3\times\nabla\right]\times\psi^\dagger\beta_1\boldsymbol{\sigma}\psi \tag{3-136}$$

which can be equivalent to Klein-Gordon if $\mu\left[\hat{\mathbf{e}}_3\times\nabla\right]\times\psi^\dagger\beta_1\boldsymbol{\sigma}\psi = -\Omega^2\psi^\dagger\boldsymbol{\sigma}\psi$.

Dirac Equation

Dirac's choice of mass term differs from the one above:

$$\partial_t\psi + c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi = -\mathrm{i}\,\Omega\beta_3\psi \tag{3-137}$$

where $M = m_e c^2/\hbar$. Other representations of this equation are:

$$\begin{gathered}\partial_t\psi + c\boldsymbol{\alpha}\cdot\nabla\psi = -\mathrm{i}\,\Omega\beta\psi \quad \text{(Dirac's original notation)}\\ \gamma^0\partial_t\psi + c\gamma^0\gamma^5\boldsymbol{\sigma}\cdot\nabla\psi \equiv \gamma^\mu\partial_\mu\psi = -\mathrm{i}\,\Omega\psi \quad \text{(Relativistic quantum mechanics notation)}\end{gathered} \tag{3-138}$$

In quantum mechanics, Planck's constant $\hbar$ appears explicitly in the operators and the wave function is normalized to one for the purpose of computing correlations. However, physically it is more sensible to normalize the wave function to $\hbar$ so that it is clear that the wave function describes the evolution of angular momentum density. One can still compute correlations, of course, as we will see later. For consistency with traditional quantum mechanics, we will include the factor of $\hbar$ in our equations.

The equation for spin angular momentum density is simply:

$$\partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = -c\nabla\left[\psi^\dagger\beta_1\psi\right] + \bar{\mathrm{i}}\,c\varepsilon_{ijk}\left[\partial_i\psi^\dagger\beta_1\sigma_k\psi - \psi^\dagger\beta_1\sigma_k\partial_i\psi\right] \tag{3-139}$$

which we interpret as an ordinary wave equation (the convection and rotation terms are presumed to cancel):

$$\partial_t^2\mathbf{Q} = c^2\nabla\left[\nabla\cdot\mathbf{Q}\right] - c^2\nabla\times\left[\nabla\times\mathbf{Q}\right] = c^2\nabla^2\mathbf{Q} \tag{3-140}$$

Dirac's choice of mass term eliminates the mass from the second-order wave equation. One consequence of this choice is that the rationale for quantization via soliton waves is lost. So while Dirac's equation can be used in describing particle motion and interactions, it cannot explain the existence of discrete particles.

Dirac also assumed that stationary states have the form:

$$\partial_t\psi = -\mathrm{i}\frac{E}{\hbar}\psi \tag{3-141}$$

which has the formal solution:

$$\psi(\mathbf{r},t) = \exp\left\{-\mathrm{i}\frac{E}{\hbar}(t-t_0)\right\}\psi(\mathbf{r},t_0) \tag{3-142}$$

This solution is puzzling because the phase variation represented by the energy eigenvalue E does not correspond to any actual oscillation in real space. The phase simply cancels out when computing observables. A more reasonable starting point would be to neglect gradients in (3-132) to get:

$$\psi(\mathbf{r},t) = \exp\left\{-\mathrm{i}\,\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}(t-t_0)\right\}\psi(\mathbf{r},t_0) \tag{3-143}$$

If the wave function is a spin eigenfunction $(\sigma_s/2)\psi = s\psi$, with eigenvalue s, then the exponent can be treated as a scalar, as in quantum mechanics. The energy eigenvalue would then represent twice the rotational energy ($E = \mathbf{w}\cdot\mathbf{S}$), consistent with an equipartition of energy between kinetic and potential energy. In this case there would also be no real oscillation. However, we can make this result sensible by assuming it to be an approximation. We suppose that the wave function is not exactly an eigenfunction of spin, so that there are oscillations in real space. For example, the spin direction may rotate at a rate small compared to the magnitude of angular velocity. For example, one can envision concentric spherical shells wobbling rigidly so that the top and bottom points from the equilibrium position rotate in circles about the z-axis, yielding a net average angular momentum. But we assume that the approximation of spin eigenfunctions is valid for the purposes of computing eigenvalues and correlations between states.

Considering the lack of real oscillation in conventional quantum mechanics, it is interesting to note that physicists in the nineteenth century, led by William Thomson (Lord Kelvin), proposed a model of vacuum as consisting of a fluid filled with vortices. This model is called the vortex sponge, and still attracts interest today. The model also has relevance to the behavior of liquid helium. This model would eliminate the requirement of oscillation, since steady flows are possible in a fluid. The model can also produce shear waves propagating among the vortices. But the model is conceptually more complex that the elastic solid, so we will not pursue it here.

If we neglect gradients in the electron equation, we have:

$$E\psi = \hbar\Omega\beta_3\psi \tag{3-144}$$

which has solutions: $\psi = [1 \quad 0 \quad 0 \quad 0]^T$ and $\psi = [0 \quad 1 \quad 0 \quad 0]^T$ for $E = \hbar\Omega$, and $\psi = [0 \quad 0 \quad 1 \quad 0]^T$ and $\psi = [0 \quad 0 \quad 0 \quad 1]^T$ for $E = -\hbar\Omega$. For each sign of E, the two solutions differ in the sign of the x_3-component of spin. These solutions are referred to as "spin-up" and "spin-down" solutions. The positive and negative signs of E are assumed to correspond to matter and anti-matter, respectively. We will now examine the relationship between matter and anti-matter further.

3.7.3. *Angular separation*

Recall Dirac's equation for a free particle:

$$\partial_t\psi = -c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi - \mathrm{i}\,\Omega\beta_3\psi \tag{3-145}$$

The operator $\boldsymbol{\sigma}\cdot\nabla\psi$ can be factored:

$$\boldsymbol{\sigma}\cdot\nabla\psi = \sigma_r\left[\partial_r + \mathrm{i}\frac{\boldsymbol{\sigma}}{r}\cdot[\mathbf{r}\times\nabla]\right]\psi = \sigma_r\left[\partial_r - \frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right]\psi \tag{3-146}$$

The two-component angular solutions of the eigenvalue equations $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi_{l,m}^{(+)} = l = -1+\kappa$ and $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi_{l,m}^{(-)} = -[l+2] = -1-\kappa$ are well known (*e.g.* (Bjorken and Drell 1964)), and are derived in Appendix A. These two angular solutions are related by $\sigma_r\Phi_{l,m}^{(+)} = \Phi_{l,m}^{(-)}$ and yield opposite eigenvalues of the parity (spatial inversion) operation.

These angular solutions may be combined to form two independent wave functions:

$$\psi^{(+)} = \frac{1}{r}\begin{bmatrix}\tilde{\mathrm{i}}\, G\Phi_{l,m}^{(+)} \\ F\Phi_{l,m}^{(-)}\end{bmatrix} \quad \text{or} \quad \psi^{(-)} = \frac{1}{r}\begin{bmatrix}\tilde{\mathrm{i}}\, F\Phi_{l,m}^{(-)} \\ G\Phi_{l,m}^{(+)}\end{bmatrix} \tag{3-147}$$

3.7.4. Velocity Rotation and Mass

It is instructive to compute the effect of mass on the wave velocity:

$$\begin{aligned}\frac{d}{dt}\left[\psi^{(+)\dagger}\beta_1\sigma_i\psi^{(+)}\right] &= \psi^{(+)\dagger}\beta_1\sigma_i\left[-\mathrm{i}\,\Omega\beta_3\psi^{(+)}\right] + \left[-\mathrm{i}\,\Omega\beta_3\psi^{(+)}\right]^{\dagger}\beta_1\sigma_i\psi^{(+)} \\ &= -2\frac{\Omega}{r^2}FG\left\{\Phi_{l,m}^{(+)\dagger}\sigma_i\Phi_{l,m}^{(-)} + \Phi_{l,m}^{(-)\dagger}\sigma_i\Phi_{l,m}^{(+)}\right\} \\ &= 2\Omega\frac{FG}{r^2}\left\{\Phi_{l,m}^{(+)\dagger}\left[\sigma_i\sigma_r + \sigma_r\sigma_i\right]\Phi_{l,m}^{(+)}\right\} \\ &= 4\Omega\frac{FG}{r^2}\delta_{ir}\left\{\Phi_{l,m}^{(+)\dagger}\Phi_{l,m}^{(+)}\right\}\end{aligned} \tag{3-148}$$

The mass term represents a radial acceleration of the wave, which is inward provided that the appropriate sign is chosen for Ω. This result implies circular propagation, consistent with the explanation of the relativistic mass-energy relation given in Chapter 1.

3.7.5. Wave Interference and Potentials

Next we investigate the origin of electromagnetic potentials. Certain observables (scalars and vectors) should be additive when two waves are superposed. This implies that when two waves ψ_A and ψ_B are superposed, the total wave ψ_T has the property that:

$$\psi_T^{\dagger}G\psi_T = \psi_A^{\dagger}G\psi_A + \psi_B^{\dagger}G\psi_B \tag{3-149}$$

for some linear Hermitian operator G. If we simply added the two wave functions, we would have instead:

$$\left[\psi_A^{\dagger} + \psi_B^{\dagger}\right]G\left[\psi_A + \psi_B\right] = \psi_A^{\dagger}G\psi_A + \psi_B^{\dagger}G\psi_B + \psi_A^{\dagger}G\psi_B + \psi_B^{\dagger}G\psi_A \tag{3-150}$$

The additional terms are clearly not zero in general. However, they can be forced to zero by introducing phase shifts to the wave functions. Using a subscript zero to represent each wave function in the absence of interference, let:

$$\psi_A = \exp\left(-\mathrm{i}\left[\delta_B + \pi\right]/2\right)\psi_{A0}$$
$$\psi_B = \exp\left(-\mathrm{i}\,\delta_A/2\right)\psi_{B0} \qquad (3\text{-}151)$$

The relative phase shift π could be distributed between the two waves or incorporated into δ_A and δ_B, but we will treat ψ_A as the 'test wave' and ψ_B as the 'source wave' and require the condition below to hold even with δ_A and δ_B equal to zero. Linear addition of the observable G requires:

$$\psi_A^{\dagger} G \psi_B + \psi_B^{\dagger} G \psi_A = 0 \qquad (3\text{-}152)$$

If either wave function is an eigenfunction of some additive observable such as spin ($G\psi_A = \lambda\psi_A$ or $G\psi_B = \lambda\psi_B$ for some scalar λ), then this result reduces to:

$$\psi_A^{\dagger}\psi_B + \psi_B^{\dagger}\psi_A = 0 \qquad (3\text{-}153)$$

In terms of the unperturbed wave functions:

$$\psi_{A0}^{\dagger} \exp\left(-\mathrm{i}\left[\pi + \delta_A - \delta_B\right]/2\right)\psi_{B0} + \psi_{B0}^{\dagger} \exp\left(\mathrm{i}\left[\pi + \delta_A - \delta_B\right]/2\right)\psi_{A0} = 0 \qquad (3\text{-}154)$$

If we interpret the quantity $\psi_A^{\dagger}\psi_B$ as a two-particle state, then interchanging the two particles yields:

$$\left\lfloor \psi_A^{\dagger}\psi_B \right\rfloor_{A \leftrightarrow B} = \psi_B^{\dagger}\psi_A = -\psi_A^{\dagger}\psi_B \qquad (3\text{-}155)$$

This means that the two-particle state is anti-symmetric with respect to exchange of particles. This symmetry is called the Pauli Exclusion Principle because it prohibits two identical fermions from being in the same state ($\psi_A = \psi_B$ yields $\psi_A^{\dagger}\psi_A = -\psi_A^{\dagger}\psi_A$). Thus the Pauli Exclusion Principle results from the arbitrary separation of the complete wave function into two independent parts. In quantum mechanics, the two-fermion state is typically constructed as:

$$\psi_{A,B} = \frac{\psi_B^{\dagger}\psi_A - \psi_A^{\dagger}\psi_B}{\sqrt{2}} \qquad (3\text{-}156)$$

so that the Exclusion Principle is automatically satisfied.

The constant phase shift $\pi/2$ has no effect on dynamics. However, some observables computed from these independent wave functions may differ from those of the free particle wave. For example:

$$\psi_{A0}^{\dagger} G \psi_{A0} = \psi_A^{\dagger}\left[\exp\left(-\mathrm{i}\,\delta_B/2\right) G \exp\left(\mathrm{i}\,\delta_B/2\right)\right]\psi_A = \psi_A^{\dagger} G_A' \psi_A \qquad (3\text{-}157)$$

Hence the effect of wave interference is to change the operator for wave packet ψ_A from G to G'_A:

$$G'_A = \exp(-\mathrm{i}\,\delta_B/2)G\exp(\mathrm{i}\,\delta_B/2) \tag{3-158}$$

Applying this rule to the operators ∂_t and H yields:

$$\{\partial_t + [\exp(-\mathrm{i}\,\delta_B/2)\partial_t \exp(\mathrm{i}\,\delta_B/2)]\}\psi_A = \exp(-\mathrm{i}\,\delta_B/2)\mathrm{i}\,H\exp(\mathrm{i}\,\delta_B/2)\psi_A \tag{3-159}$$

$$\frac{H}{\hbar} = \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla - \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2} \tag{3-160}$$

Substituting the general form of the Hamiltonian and allowing for convection:

$$\begin{gathered}\left[\partial_t + \frac{\mathrm{i}}{2}\partial_t\delta_B\right]\psi'_B + \left[c\beta_1\boldsymbol{\sigma}\cdot\nabla + \frac{\mathrm{i}}{2}c\beta_1\boldsymbol{\sigma}\cdot\nabla\delta_B\right]\psi'_B \\ +\left[\mathbf{u}_A\cdot\nabla + \frac{\mathrm{i}}{2}\mathbf{u}_A\cdot\nabla\delta_B\right]\psi'_B + \mathrm{i}(\mathbf{w}_A + \mathbf{w}_B)\cdot\frac{\boldsymbol{\sigma}}{2}\psi'_B = 0\end{gathered} \tag{3-161}$$

Substituting the mass term for the free electron:

$$\begin{gathered}\left[\partial_t + \frac{\mathrm{i}}{2}\partial_t\delta_B\right]\psi_A + \left[c\beta_1\boldsymbol{\sigma}\cdot\nabla + \frac{\mathrm{i}}{2}c\beta_1\boldsymbol{\sigma}\cdot\nabla\delta_B\right]\psi_A \\ +\left[\mathbf{u}_B\cdot\nabla + \frac{\mathrm{i}}{2}\mathbf{u}_A\cdot\nabla\delta_B\right]\psi_A + \frac{\mathrm{i}}{2}[\boldsymbol{\sigma}\cdot\mathbf{w}_B]\psi_A + \mathrm{i}\Omega\beta_3\psi_A = 0\end{gathered} \tag{3-162}$$

Since we are interested in the effects of the phase shift, we will neglect the extra terms which are independent of δ_B (without explicit justification). We then define the electromagnetic potentials as:

$$\begin{aligned}\frac{e}{\hbar}\mathbf{A} &\equiv -\frac{c}{2}\nabla\delta_B \\ \frac{e}{\hbar}\boldsymbol{\Phi} &\equiv \frac{1}{2}\partial_t\delta_B - \mathbf{u}\cdot\frac{e}{\hbar c}\mathbf{A}\end{aligned} \tag{3-163}$$

Although the vector potential **A** is a gradient, its curl (the magnetic field) may be nonzero because δ_B is a phase angle which may be multi-valued. For example, the multi-valued function $\delta_B = \arctan(x_2/x_1)$ has gradient components:

$$\begin{aligned}\partial_1\delta_B &= -\frac{x_2}{\left[x_1^2 + x_2^2\right]^{(1/2)}} \\ \partial_2\delta_B &= \frac{x_1}{\left[x_1^2 + x_2^2\right]^{(1/2)}}\end{aligned} \tag{3-164}$$

The curl of this gradient is non-zero at $(x_1,x_2)=(0,0)$. See Kleinert [2007] for a discussion of multi-valued potentials in electromagnetism.

With these definitions, the electron equation in the presence of another wave becomes:

$$\left[\partial_t + \mathrm{i}\frac{e}{\hbar}\Phi\right]\psi_A + \left[c\beta_1\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\frac{e}{\hbar}\beta_1\boldsymbol{\sigma}\cdot\mathbf{A}\right]\psi_A + \mathrm{i}\,\Omega\beta_3\psi_A = 0 \tag{3-165}$$

Hence electromagnetic potentials result from wave interference under the assumption that different wave packets are independent. The above analysis is not very precise, however, as we neglected changes in medium velocity and vorticity, and did not specify which observables should be additive (total momentum density and total angular momentum density should both have this property). A complete analysis of particle interactions would require knowledge of the soliton wave functions of each particle.

Setting $\hbar\partial_t\psi_A = \mathrm{i}\,H\psi$, the modified Hamiltonian is:

$$H = -e\Phi + \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla + c\beta_1\boldsymbol{\sigma}\cdot\frac{e}{c}\mathbf{A} - \Omega\beta_3 \tag{3-166}$$

Multiple source waves may be treated sequentially, at least as a first approximation. For a given test wave, make it independent of the first source wave as above. Then take the modified test wave and make it independent of the second source wave. Repetition of this process for all source waves results in the addition of phase shifts or equivalently, the addition of potentials. Matter and anti-matter solutions are assumed to yield opposite signs of phase shift. One may also infer that soliton waves with identical long-range (electromagnetic) potentials (e.g. positrons and protons) also have identical bispinor wave functions at large distances from their centers.

In quantum mechanics, it is necessary to treat various wave packets as independent 'particles'. However, with a classical wave theory of matter it may be simpler to solve the single equation for the total angular momentum density, then decompose the solution into soliton 'particles' for comparison with experiment.

3.7.6. Lorentz Force

In terms of electromagnetic potentials, the modified Hamiltonian is:

$$\frac{H}{\hbar} = \bar{q}\Phi - \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla - \beta_1\boldsymbol{\sigma}\cdot\bar{q}\mathbf{A} - \mathrm{i}\,\mathbf{u}\cdot\nabla + \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2} = \bar{q}\Phi - \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla - \beta_1\boldsymbol{\sigma}\cdot\bar{q}\mathbf{A} + \Omega\beta_3 \tag{3-167}$$

Recalling the u-dependence of $q\Phi$ and $\mathbf{w}$ (and our change of sign of H), the conjugate momentum for $\mathbf{r}$ is now:

$$\mathbf{p_r} = \frac{\delta L}{\delta[\mathbf{u}]} = \frac{\delta\,\mathrm{H}}{\delta[\mathbf{u}]} = -\psi^\dagger\left\{\mathrm{i}\nabla + \frac{q}{c}\mathbf{A}\right\}\psi + \rho\mathbf{u} = \mathbf{p}_0 - \frac{q}{c}\mathbf{A} + \mathbf{q} \tag{3-168}$$

where $\mathbf{p}_0 = \psi^\dagger\{-\mathrm{i}\nabla\}\psi$ is the free particle wave momentum.

The time derivative of any observable Q is:

$$\partial_t\left[\psi^\dagger Q\psi\right] = \left[\partial_t\psi^\dagger\right]Q\psi + \psi^\dagger Q\partial_t\psi + \psi^\dagger\left[\partial_t Q\right]\psi = \psi^\dagger\,\mathrm{i}\left[H,Q\right]\psi + \psi^\dagger\partial_t Q\psi \tag{3-169}$$

An example of this is the force density. Substituting the linear wave momentum for Q yields the Lorentz force law:

$$\partial_t \mathbf{p} = \psi^\dagger \left\{ \nabla \left[c\beta_1 \boldsymbol{\sigma} \cdot \frac{q}{c} \mathbf{A} \right] - \nabla q\Phi - c\beta_1 \boldsymbol{\sigma} \cdot \nabla \left[\frac{q}{c} \mathbf{A} \right] - \frac{q}{c} \frac{\partial}{\partial t} \mathbf{A} \right\} \psi$$

$$= \psi^\dagger \left\{ c\beta_1 \boldsymbol{\sigma} \times \left[\nabla \times \frac{q}{c} \mathbf{A} \right] - \nabla q\Phi - \frac{q}{c} \frac{\partial}{\partial t} \mathbf{A} \right\} \psi = \psi^\dagger \left\{ c\beta_1 \boldsymbol{\sigma} \times \frac{q}{c} \mathbf{B} + q\mathbf{E} \right\} \psi \tag{3-170}$$

where **E** and **B** are the usual electric and magnetic fields, respectively. Hence the Lorentz force has a straightforward interpretation in terms of classical wave interference.

3.7.7. Magnetic Moment

The equation of evolution in electromagnetic fields is:

$$\left[\partial_t + \mathrm{i} q\Phi + c\beta_1 \boldsymbol{\sigma} \cdot \nabla + \mathrm{i} q\beta_1 \boldsymbol{\sigma} \cdot \mathbf{A} \right] \psi = -\mathrm{i}\Omega\beta_3 \psi \tag{3-171}$$

Using two-component spinors with $\psi = [\psi_1, \psi_2]^T$, this equation can be separated into two coupled equations:

$$\begin{aligned} \left[\partial_t + \mathrm{i} q\Phi \right] \psi_1 + \left[c\boldsymbol{\sigma} \cdot \nabla - \mathrm{i} q\boldsymbol{\sigma} \cdot \mathbf{A} \right] \psi_2 &= -\mathrm{i}\Omega \psi_1 \\ \left[\partial_t + \mathrm{i} q\Phi \right] \psi_2 + \left[c\boldsymbol{\sigma} \cdot \nabla - \mathrm{i} q\boldsymbol{\sigma} \cdot \mathbf{A} \right] \psi_1 &= \mathrm{i}\Omega \psi_2 \end{aligned} \tag{3-172}$$

Let $\psi_1 \equiv \exp(-i\Omega t)\chi_1$ and $\psi_2 = \exp(-i\Omega t)\chi_2$. Substitution yields:

$$\begin{aligned} \left[\partial_t + \mathrm{i} q\Phi \right] \chi_1 + \left[c\boldsymbol{\sigma} \cdot \nabla - \mathrm{i} q\boldsymbol{\sigma} \cdot \mathbf{A} \right] \chi_2 &= 0 \\ \left[\partial_t + \mathrm{i} q\Phi - 2\mathrm{i} M \right] \chi_2 + \left[c\boldsymbol{\sigma} \cdot \nabla - \mathrm{i} q\boldsymbol{\sigma} \cdot \mathbf{A} \right] \chi_1 &= 0 \end{aligned} \tag{3-173}$$

Next, assume that $\left| \left[\partial_t + \mathrm{i}\bar{q}\Phi \right] \chi_2 \right| << \left| 2\mathrm{i}\Omega\chi_2 \right|$. This yields:

$$\left[\partial_t + \mathrm{i} q\Phi \right] \chi_1 + \frac{\left[c\boldsymbol{\sigma} \cdot \nabla - \mathrm{i} q\boldsymbol{\sigma} \cdot \mathbf{A} \right] \left[c\boldsymbol{\sigma} \cdot \nabla - \mathrm{i} q\boldsymbol{\sigma} \cdot \mathbf{A} \right]}{2\mathrm{i}\Omega} \chi_1 = 0 \tag{3-174}$$

This is the Pauli equation, which was the first equation to incorporate electron spin.

$$\mathrm{i}\partial_t \chi_1 = \left\{ \frac{\boldsymbol{\sigma} \cdot \left[\mathrm{i} c\nabla - q\mathbf{A} \right] \boldsymbol{\sigma} \cdot \left[\mathrm{i} c\nabla - q\mathbf{A} \right]}{2\Omega} + q\Phi \right\} \chi_1 \tag{3-175}$$

Using the commutation relations for the Pauli spin matrices:

$$\begin{aligned} \boldsymbol{\sigma} \cdot \left[-\mathrm{i} c\nabla - q\mathbf{A} \right] \boldsymbol{\sigma} \cdot \left[\mathrm{i} c\nabla - q\mathbf{A} \right] \chi_1 &= \sigma_i \left[-\mathrm{i} c\partial_i - qA_i \right] \sigma_j \left[-\mathrm{i} c\partial_j - qA_j \right] \chi_1 \\ &= \left\{ \left[-\mathrm{i} c\partial_i - qA_i \right]^2 + \varepsilon_{ijk}\, \mathrm{i}\sigma_k \left[-\mathrm{i} c\partial_i - qA_i \right] \left[-\mathrm{i} c\partial_j - qA_j \right] \right\} \chi_1 \\ &= \left\{ \left[-\mathrm{i} c\partial_i - qA_i \right]^2 - cq\boldsymbol{\sigma} \cdot \mathbf{B} \right\} \chi_1 \end{aligned} \tag{3-176}$$

Substitution yields:

$$\mathrm{i}\hbar\partial_t \chi_1 = \hbar \left\{ \frac{\left[-\mathrm{i} c\partial_i - qA_i \right]^2 - c\bar{q}\boldsymbol{\sigma} \cdot \mathbf{B}}{2\Omega} + q\Phi \right\} \chi_1 \tag{3-177}$$

This equation is of course simply an approximate equation for two components of the Dirac wave function. Nonetheless, it is of historical importance because it was used by Pauli to include effects of electron spin. Without the spin term, the resultant scalar equation is the one Schrödinger first used to compute the hydrogen energy levels:

$$\mathrm{i}\hbar\partial_t\psi = \hbar\left\{\frac{\left[-\mathrm{i}c\partial_i - qA_i\right]^2}{2\Omega} + (q/\hbar)\Phi\right\}\psi = \left\{\frac{p^2}{2mc^2} + q\Phi\right\}\psi \tag{3-178}$$

Schrödinger's equation is currently the conventional starting point in the study of quantum mechanics. Although simpler than the Dirac equation, it is far less intuitive. Both Lorentz invariance and the connection with spin angular momentum have been lost.

In a weak, uniform magnetic field with $\mathbf{A} = \mathbf{B}_0 \times \mathbf{r}/2$, we can neglect A^2 to obtain:

$$\begin{aligned}\mathrm{i}\hbar\partial_t\chi_1 &= \left\{\hbar\frac{-c^2\nabla^2 + \mathrm{i}cq\left[\mathbf{B}_0\times\mathbf{r}\right]\cdot\nabla - cq\boldsymbol{\sigma}\cdot\mathbf{B}_0}{2\Omega} + q\Phi\right\}\chi_1 \\ &= \left\{\hbar\frac{-c^2\nabla^2 - cq\mathbf{B}_0\cdot\left(\left[\mathbf{r}\times -\mathrm{i}\nabla\right] + \boldsymbol{\sigma}\right)}{2\Omega} + q\Phi\right\}\chi_1 = \left\{-\frac{\hbar^2}{2m}\nabla^2 - \frac{\hbar q}{2mc}\mathbf{B}_0\cdot\left[\mathbf{L} + 2\mathbf{s}\right] + q\Phi\right\}\chi_1\end{aligned} \tag{3-179}$$

The final form with the spin angular momentum operator ($\mathbf{s} = \boldsymbol{\sigma}/2$) is obtained by comparison with the angular momentum operator (3-**Error! Bookmark not defined.**). This result is significant because it shows that, in this approximation, the coefficient of spin angular momentum is twice the coefficient of orbital angular momentum in the electron magnetic moment:

$$\boldsymbol{\mu} = -\frac{cq}{2\Omega}\left[\mathbf{L} + 2\mathbf{S}\right] \tag{3-180}$$

A free electron with $q = -e$, $\mathbf{L} = 0$, and $|\mathbf{S}| = 1/2$, has magnetic moment equal (within 0.1%) to the Bohr magneton $e\hbar/2mc = 5.78\times10^{-5}$ eV/T.

3.7.8. Spin Waves

Consider the equation for the evolution of spin (3-109):

$$\partial_t\mathbf{S} - c^2\nabla^2\mathbf{Q} + \mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{w}\times\mathbf{S} = 0 \tag{3-181}$$

If we neglect the spatial gradients, we have:

$$\partial_t\mathbf{S} = \mathbf{w}\times\mathbf{S} \tag{3-182}$$

The vorticity is given by:

$$\mathbf{w} = \frac{1}{2}\nabla\times\mathbf{u} = \frac{1}{2\rho}\nabla\times\left[-\mathrm{Re}\left(\psi^\dagger\,\mathrm{i}\nabla\psi\right) + \frac{1}{2}\nabla\times\mathbf{S}\right] \tag{3-183}$$

Keeping only the term involving spin yields:

$$\partial_t\mathbf{S} = \frac{1}{4\rho}\nabla\times\left[\nabla\times\mathbf{S}\right]\times\mathbf{S} = -\frac{1}{4\rho}\nabla^2\mathbf{S}\times\mathbf{S} \tag{3-184}$$

This equation describes the simplest form of a 'spin wave', which is commonly observed in ferromagnetic materials.

3.7.9. Measurement Correlations

In 1935 Einstein, Podolsky, and Rosen suggested a thought experiment intended to demonstrate that quantum mechanics was not a complete theory. The idea was that particles generated in pairs could be subjected to independent measurements that are not quantum mechanically allowed on single particles (due *e.g.* to the uncertainty principle). However, when actual experiments were performed they supported the quantum mechanical view that physical quantities do not have specific values until they are measured.

It is widely believed that the correlations between polarization measurements of entangled particles cannot be predicted classically. This belief is based on correlation predictions using an equation of the form:

$$P(\mathbf{a},\mathbf{b}) = \int A(\mathbf{a},\lambda_1,\ldots,\lambda_n)B(\mathbf{b},\lambda_1,\ldots,\lambda_n)\rho(\lambda_1,\ldots,\lambda_n)d\lambda_1..d\lambda_n \tag{3-185}$$

where λ_i represent variables which describe the state of the system, $\rho(\lambda_1,\ldots,\lambda_n)$ is the probability distribution of these variables, **a** and **b** are the measured polarization directions for the two entangled particles, A and B are the theoretical outcomes of the measurement (±1), and $P(\mathbf{a},\mathbf{b})$ is the correlation.

John Bell [1964] proved that quantum correlations cannot be represented in this form. In particular, he proved that for three different measurements $A(\mathbf{a},\lambda_1,\ldots,\lambda_n)$, $B(\mathbf{b},\lambda_1,\ldots,\lambda_n)$, and $C(\mathbf{c},\lambda_1,\ldots,\lambda_n)$:

$$1 + P(\mathbf{b},\mathbf{c}) \geq |P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \tag{3-186}$$

This condition is violated by quantum mechanical (and physically observed) correlations, which can be measured using two or more particles whose spins are constrained. For example, if a pair of spin ½ particles is produced with opposite spin, the correlation between their spin measurements by detectors oriented with relative angle φ is:

$$P_{\text{pair}}(\varphi) = -\cos\varphi \tag{3-187}$$

This correlation violates Bell's condition. For example, if the detectors **a**, **b**, and **c** are oriented at angles 0, $\pi/4$, and $3\pi/4$, respectively, then:

$$\begin{aligned} P(\mathbf{a},\mathbf{b}) &= P(\pi/4) = -1/\sqrt{2} \\ P(\mathbf{b},\mathbf{c}) &= P(\pi/2) = 0 \\ P(\mathbf{a},\mathbf{c}) &= P(3\pi/4) = 1/\sqrt{2} \\ 1 + P(\mathbf{b},\mathbf{c}) &< |P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \end{aligned} \tag{3-188}$$

The fact that actual measurements violate Bell's condition implies that some aspect of his derivation does not conform to reality. One explicit assumption is that there is no communication, instantaneous or otherwise, between the two detectors. Many have interpreted violation of Bell's inequalilty as evidence of *instantaneous* communication between the detectors (nonlocality), but there is no direct evidence that information can be transmitted instantaneously.

We will leave this issue unresolved, but point out that the integral (3-185) is questionable. For example, if some parameters are complex numbers then the integral is contour-dependent and may be ill-defined. Furthermore, the wave model of matter is also nonlocal insofar as each "particle" is actually a spatially extended wave packet.

An alternative formulation utilizes conditional probabilities. Let $A_i B_j$ represent the four possible measurement outcomes at detectors A and B: A_+B_+, A_-B_-, A_+B_-, and A_-B_+. Defining $\Pr(x)$ as the probability of *x*, the correlation may be written as:

$$P(\mathbf{a},\mathbf{b}) = \sum_{i,j} A_i B_j \Pr(A_i \cap B_j) = \sum_{i,j} A_i B_j \Pr(A_i)\Pr(B_j \mid A_i) = \frac{1}{2}\sum_{i,j} A_i B_j \Pr(B_j \mid A_i) \tag{3-189}$$

We will use this form and assume that the conditional probability $\Pr(B_j \mid A_i) = \Pr(A_i \mid B_j)$ is proportional to the squared correlation between spinor eigenfunctions separated by the rotation angle between the measured states.

To compute the correlation between two bispinor wave functions, consider the following properties: First, the magnitude of the wave function must be second-order in each of the components and positive-definite. Therefore:

$$\|\psi\| = \psi^\dagger \psi \tag{3-190}$$

Second, physical variables are bilinear in the wave function. Therefore it is the squared magnitude that is of physical interest. The un-normalized correlation P_0 between two functions must be defined in such a way that the squared norm $\left|\psi^\dagger\psi\right|^2$ is the self-correlation:

$$P_0(\psi_A, \psi_B) = \left|\psi_A^\dagger \psi_B\right|^2 \tag{3-191}$$

Dividing by the magnitudes of each wave function yields the normalized correlation *C*:

$$P(\psi_A, \psi_B) = \frac{P_0(\psi_A, \psi_B)}{\left|\psi_A^\dagger \psi_A\right|\left|\psi_B^\dagger \psi_B\right|} = \frac{\left|\psi_A^\dagger \psi_B\right|^2}{\left|\psi_A^\dagger \psi_A\right|\left|\psi_B^\dagger \psi_B\right|} \tag{3-192}$$

The correlation between states related by rotation $R(\boldsymbol{\varphi})$ about an axis perpendicular to the spin is:

$$P = \frac{\left|\psi^\dagger R(\boldsymbol{\varphi})\psi\right|^2}{\left|\psi^\dagger\psi\right|^2} = \cos^2\frac{\varphi}{2} \tag{3-193}$$

The correlation for angle $(\pi\hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi})$ is $\cos^2[(\pi-\varphi)/2] = \sin^2[\varphi/2]$.

Assuming that spin measurements are coincident or anti-coincident in proportion to the correlations between the spinor wave functions, the correlation P_s between spin measurements separated by angle $\boldsymbol{\varphi}$ is:

$$P_s(\varphi) = P_\psi(\varphi) - P_\psi(\pi - \varphi) = \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2} = \cos\varphi \tag{3-194}$$

In the case of pair production in EPR-type experiments, the spins of the two particles are opposite (changing φ to $\pi - \varphi$ above), thereby changing the sign of the correlation. Hence we are able to derive the quantum correlations from some simple assumptions, though this is by no means a definitive resolution of the EPR paradox.

3.7.10. Quantum Mechanics

In the preceding section we computed the correlation between two states related by rotation. The two states may be denoted by $\psi(\mathbf{r},t)$ and $R(\boldsymbol{\varphi})\psi(\mathbf{r},t)$. The correlation at a given position and time is given by (3-193). A more global correlation between two wave functions $\psi_1(\mathbf{r},t)$ and $\psi_2(\mathbf{r},t)$ at a given time is obtained by integrating over space:

$$P = \frac{\left|\int \psi_1^\dagger \psi_2\, d^3r\right|^2}{\left|\int \psi_1^\dagger \psi_1\, d^3r\right|\left|\int \psi_2^\dagger \psi_2\, d^3r\right|} = \frac{\left|\int \psi_2^\dagger \psi_1\, d^3r\right|^2}{\left|\int \psi_1^\dagger \psi_1\, d^3r\right|\left|\int \psi_2^\dagger \psi_2\, d^3r\right|} \tag{3-195}$$

The correlation between spin one-half states is non-negative, and the correlation of a wave function with itself is unity. These properties provide the basis for a probabilistic interpretation of the wave functions. A given wave function may be decomposed into multiple wave functions (states), and the correlation between the wave function and each 'state' may be computed. In quantum mechanics, this correlation is interpreted as the probability of detecting that state with a measurement.

This means that correlations between physical states (as opposed to measurements) are equal to the square of a complex amplitude. This fundamental property of quantum mechanics has mystified generations of physicists. Yet we can now see clearly that this property of matter is due to the simple fact that independent wave states are 180 degrees apart.

Temporal evolution of the wave function is expressed as:

$$\psi(\mathbf{r},t_2) = \exp\left(-\mathrm{i}\int_{t_1}^{t_2} H(\mathbf{r},t)dt\right)\psi(\mathbf{r},t_1) \tag{3-196}$$

Therefore the correlation between an initial state $\psi_1(\mathbf{r},t_1)$ and a final state and $\psi_2(\mathbf{r},t_2)$ is:

$$P(\psi_2(t_2) \mid \psi_1(t_1)) = \frac{\left|\int \psi_2^\dagger \exp\left(-\mathrm{i}\int_{t_1}^{t_2} H(\mathbf{r},t)dt\right)\psi_1\, d^3r\right|^2}{\left|\int \psi_1^\dagger \psi_1\, d^3r\right|\left|\int \psi_2^\dagger \psi_2\, d^3r\right|} \tag{3-197}$$

In quantum mechanics, the states are normalized to one:

$$\psi' = \frac{\psi}{\left|\int \psi^{\dagger} \psi \, d^3 r\right|^{1/2}} \tag{3-198}$$

Dropping the primes, the correlation integrals are then written in the form:

$$P(\psi_2(t_2) \mid \psi_1(t_1)) \equiv \left|\langle \psi_2 \| \psi_1 \rangle\right|^2 = \left|\langle \psi_1 \right| \exp\left(-\mathrm{i}\int_{t_1}^{t_2} H(\mathbf{r},t)\,dt\right)\left|\psi_1\rangle\right|^2 \tag{3-199}$$

In quantum mechanics, this correlation represents the probability density for the initial state ψ_1 to evolve into the final state ψ_2.

Any physical process can be analyzed statistically in terms of complex amplitudes such as described above. If different possible final states are distinguishable, then the joint probability is obtained simply be adding each of the separate probabilities. However, if different possible final states are indistinguishable (e.g. one electron or a different electron reaching a detector), then the joint probability is computed by adding the amplitudes and only then computing the magnitude. In the wave model, this rule is explained by the fact that indistinguishable particles (e.g. two electrons) are wave packets with the same frequency characteristics. Non-identical particles, which have different frequency characteristics, have variable phase between the two waves and therefore any interference between the two waves would average to zero.

3.7.11. Fermions and Bosons

Particles whose correlations are computed according to the above rules are called "fermions" in honor of the physicist Enrico Fermi. Fermions are considered to be the "fundamental particles" of nature. These include electrons, protons, neutrons, neutrinos, and quarks. Recall that the Pauli exclusion principle was derived from the assumption that the particle wave functions were eigenfunctions of an observable (e.g. spin). If this is not the case, then there is no exclusion principle.

Particles which can be superposed are called "bosons" in honor of physicist Satyendranath Bose. Examples include photons and π mesons. Multiple bosons may coexist with each in exactly the same state (and same position). In quantum mechanics the boson two-particle wave function satisfies:

$$\psi_A^{\dagger}\psi_B - \psi_B^{\dagger}\psi_A = 0 \tag{3-200}$$

This condition is always satisfied if $\psi_A = \psi_B$, so there is no exclusion principle for bosons.

To see how spin is related to statistics, consider a massless photon which in the plane wave approximation satisfies the equation:

$$\partial_t^2 \mathbf{Q} - c^2 \nabla^2 \mathbf{Q} = \left(\partial_t - c\hat{\mathbf{k}}\cdot\nabla\right)\left(\partial_t + c\hat{\mathbf{k}}\cdot\nabla\right)\mathbf{Q} \tag{3-201}$$

Either $\left(\partial_t + c\hat{\mathbf{k}}\cdot\nabla\right)\mathbf{Q} = 0$ or $\left(\partial_t - c\hat{\mathbf{k}}\cdot\nabla\right)\mathbf{Q} = 0$. In either case the vector **Q** obeys a convection equation and is therefore the quantity used to compute correlations. **Q** is a vector, which transforms under rotation with spin one. Multiple photons can be superposed simply by adding their **Q** values without the interference associated with spinors.

For another example, suppose fermions A and B are somehow bound together with a joint wave function $\psi_{A,B}$ which satisfies the exclusion principle:

$$\psi_{A,B} = \frac{\psi_B^\dagger \psi_A - \psi_A^\dagger \psi_B}{\sqrt{2}} \tag{3-202}$$

If we use $\psi_{A,B}$ to compute correlations with an identical particle composed of fermions A' and B', we have:

$$\begin{aligned}&\psi_{A,B}^\dagger \psi_{A',B'} - \psi_{A',B'}^\dagger \psi_{A,B} \\ &= \left[\frac{\psi_A^\dagger \psi_B - \psi_B^\dagger \psi_A}{\sqrt{2}}\right]\left[\frac{\psi_{B'}^\dagger \psi_{A'} - \psi_{A'}^\dagger \psi_{B'}}{\sqrt{2}}\right] - \left[\frac{\psi_{A'}^\dagger \psi_{B'} - \psi_{B'}^\dagger \psi_{A'}}{\sqrt{2}}\right]\left[\frac{\psi_B^\dagger \psi_A - \psi_A^\dagger \psi_B}{\sqrt{2}}\right] = 0\end{aligned} \tag{3-203}$$

Hence composite particles formed from two fermions behave statistically like bosons.

In the Standard Model of Physics, the viewpoint is that fundamental particles are fermions which interact through fields, and particles associated with the fields are bosons. We have seen how this interpretation can arise from an underlying classical wave process.

3.7.12. Prior Knowledge and Statistics

Interpretation of quantum statistics can be confusing. Consider the case of Schrodinger's cat. The cat is placed in a box which contains a radioactive element, a radiation detector, and a poisonous gas. If the detector is triggered by a radioactive decay then it will in turn trigger the release of the poison and thereby kill the cat. According to quantum statistics, at any given time there is not merely a chance that the cat will be dead or alive, but the mathematical description involves a complex amplitude for each possibility. Just as electron statistics were described above by a complex superposition of 'spin up' and 'spin down' states, the cat's fate is described by a complex superposition of 'alive' and 'dead' states. Physicists are therefore tempted to say that the cat is in a superposition of living and dead states, which is rather absurd.

There are different ways to resolve this paradox, but the simplest resolution is to say that the cat really is either dead or alive, and not both. The complex amplitude merely indicates our knowledge (or lack of knowledge) of the situation. Physicists have generally rejected this logic because they never realized that classical statistics (e.g. the probability that the cat is dead) should be computed in exactly the same manner as the quantum statistics. The Copenhagen interpretation of quantum mechanics posits that the statistical interpretation of the complex wave function is also the physical interpretation (i.e. there are no deterministic physical variables because if there were then their correlations would be computed differently). However, we can obtain the same correlations without the bizarre interpretation that the cat is partly alive and partly dead until we open the box.

3.7.13. Hydrogen Atom

The proton produces a Coulomb potential ($e\Phi = -Ze^2/r$). Neglecting the vector potential in the electromagnetic electron equation (3-165) yields:

$$\left[\partial_t + \mathrm{i}\frac{e}{\hbar}\Phi + c\beta_1\boldsymbol{\sigma}\cdot\nabla\right]\psi = -\mathrm{i}\,\Omega\beta_3\psi \tag{3-204}$$

Assume as before a temporal eigenvalue $\partial_t\psi = -\mathrm{i}\,E\psi$, and assume that the angular eigenfunction $\Phi_{l,m}^{(+)}$ has even parity and $\Phi_{l,m}^{(-)}$ odd parity. A wave function of the form $\psi^{(+)}$ yields the coupled radial equations:

$$\begin{aligned}\left[E - \frac{e}{\hbar}\Phi - \Omega\right]G + c\left[\partial_r + \frac{\kappa}{r}\right]F = 0\\ \left[E - \frac{e}{\hbar}\Phi + \Omega\right]F - c\left[\partial_r - \frac{\kappa}{r}\right]G = 0\end{aligned} \tag{3-205}$$

Solutions to these coupled equations are obtained as follows (e.g. Schiff 1968):

For large r the asymptotic equations are:

$$\begin{aligned}[E - \Omega]G + c\partial_r F = 0\\ [E + \Omega]F - c\partial_r G = 0\end{aligned} \tag{3-206}$$

which combine to yield:

$$\left[E^2 - \Omega^2\right]F + c^2\partial_r^2 F = 0 \tag{3-207}$$

We are seeking a bound state with $E^2 < \Omega^2$. Therefore the asymptotic behavior is $F \propto \exp(-\alpha r)$ with $\alpha \equiv \sqrt{\left[\Omega^2 - E\right]/c^2}$.

Now let:

$$\begin{aligned}F(r) = f(r)\exp(-\alpha r)\\ G(r) = g(r)\exp(-\alpha r)\end{aligned} \tag{3-208}$$

The coupled equations become:

$$\begin{aligned}\left[E - \frac{e}{\hbar}\Phi - \Omega\right]g + c\left[\partial_r - \alpha + \frac{\kappa}{r}\right]f = 0\\ \left[E - \frac{e}{\hbar}\Phi + \Omega\right]f - c\left[\partial_r - \alpha - \frac{\kappa}{r}\right]g = 0\end{aligned} \tag{3-209}$$

Assume that f and g can be written as power series:

$$f(r) = \sum_{\nu=0}^{\infty} f_\nu r^{s+\nu}$$

$$g(r) = \sum_{\nu=0}^{\infty} g_\nu r^{s+\nu} \tag{3-210}$$

Let $\gamma \equiv Ze^2/\hbar c$ and match powers of r:

$$\begin{aligned}[E-\Omega]g_{\nu-1} - c\alpha f_{\nu-1} + \gamma c g_\nu + c[s+\nu+\kappa]f_\nu &= 0\\ [E+\Omega]f_{\nu-1} + c\alpha g_{\nu-1} + \gamma c f_\nu - c[s+\nu-\kappa]g_\nu &= 0\end{aligned} \tag{3-211}$$

We can eliminate the $(\nu-1)$ terms to get a relationship between f_ν and g_ν.

$$\{[E-\Omega]c[s+\nu-\kappa] + c^2\alpha\gamma\}g_\nu = \{[E-\Omega]\gamma c - c^2\alpha[s+\nu+\kappa]\}f_\nu \tag{3-212}$$

which for large n becomes $[E-\Omega]g_\nu = -c\alpha f_\nu$.

For n=0:

$$\begin{aligned}\gamma g_0 + [s+\kappa]f_0 &= 0\\ \gamma f_0 - [s-\kappa]g_0 &= 0\end{aligned} \tag{3-213}$$

The determinant for these coupled equations must be zero. This condition yields a solution for s:

$$s = \pm\sqrt{\kappa^2 - \gamma^2} \tag{3-214}$$

Recall that the actual wave function contains an additional factor of 1/r. Therefore we choose the positive sign here so that the solution is regular (or only slightly divergent if $|s|<1$) at the origin.

Using the relation between coefficients derived above, the asymptotic behavior for large ν is:

$$\begin{aligned}-2\alpha f_{\nu-1} + \nu f_\nu &= 0\\ 2\alpha g_{\nu-1} - \nu g_\nu &= 0\end{aligned} \tag{3-215}$$

The ratio between successive terms matches the Taylor series expansion for exp(2ar):

$$\exp(2\alpha r) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}[2\alpha r]^\nu \tag{3-216}$$

If the series proceeds to infinite n then the wave function would be infinite at large values of r. To make the wave function finite, the series must terminate at some finite value of ν. Calling this value n', Eq. (3-211) yields the relation between the highest coefficients:

$$c\alpha g_{n'} = -[E+\Omega]f_{n'} \tag{3-217}$$

Combining this relation with eq. (3-212) yields an expression for the characteristic frequencies:

$$E\gamma = c^2[s+n']\alpha = c^2[s+n'][\Omega^2 - E^2]^{1/2} \tag{3-218}$$

Solving for $\hbar E$:

$$\hbar E = \hbar\Omega\left\{1 + \gamma^2 / [s + n']^2\right\}^{-1/2} \tag{3-219}$$

These are the discrete energy levels of an electron in a Coulomb potential. The factor of $\hbar$, which relates energy and frequency, is assumed to be the integral of the squared wave function. Denote the energy by $\mathcal{E} \equiv \hbar E$ and mass by $m_e c^2 \equiv \hbar\Omega$. These energy levels were actually derived by Sommerfeld [1916a] using the model of a relativistic particle propagating in elliptical orbits.

There are two main sources of discrepancy from the actual hydrogen energy levels. First, we assumed a static potential, implying that the nucleus is unaffected by the presence of the electron wave. By analogy with particles we can improve the calculations by replacing the electron rest energy $\hbar\Omega = m_e c^2$ with the "reduced mass" energy $\hbar\Omega' = c^2 m_e m_p / [m_e + m_p]$, where m_p is the proton mass. Second, we have neglected any effects of the magnetic vector potential.

The energy levels are typically classified using a positive integer principal quantum number *n* and positive half-integer angular quantum number $J \equiv \kappa - 1/2$:

$$n = J + \frac{1}{2} + n' \tag{3-220}$$

In terms of these quantum numbers the energy levels are:

$$\mathcal{E} = \hbar\Omega'\left\{1 + \frac{\gamma^2}{\left[n - J - 1/2 + \sqrt{[J + 1/2]^2 - \gamma^2}\right]^2}\right\}^{-1/2} \tag{3-221}$$

Table 3-II compares measured energy levels (relative to the ground state) with energy levels calculated using this formula. The configuration label (*nL*) includes the principal quantum number *n* followed by a letter code for the orbital angular momentum *L*: s=0, p=1, d=2, f=3, etc. Note that the formula above does not distinguish between different *L* values for the same *n* and *J*.

While the agreement with experiment is good, it must be noted that the assumed Coulomb potential is simply empirical (as it is also in conventional quantum theory). For a complete theory the potentials of the nucleus should be derived from its free particle wave function.

Configuration	*J*	Measured Level (eV)	Level Computed from (3-221)
1s	1/2	0	0
2s	1/2	10.1988101	10.1988390
2p	1/2	10.1988057	10.1988390
2p	3/2	10.1988511	10.1988843
3s	1/2	12.0874944	12.0875263
3p	1/2	12.0874931	12.0875263

3p	3/2	12.0875066	12.0875397
3d	3/2	12.0875065	12.0875397
3d	5/2	12.0875110	12.0875442
4s	1/2	12.7485324	12.7485650
4p	1/2	12.7485319	12.7485650
4p	3/2	12.7485375	12.7485707
4d	3/2	12.7485375	12.7485707
4d	5/2	12.7485394	12.7485726
4f	5/2	12.7485394	12.7485726
4f	7/2	12.7485404	12.7485735
$n \to \infty$		13.5984340	13.5984671

Table 3-II Measured and computed hydrogen energy levels.

Ralchenko, Yu., Jou, F.-C., Kelleher, D.E., Kramida, A.E., Musgrove, A., Reader, J., Wiese, W.L., and Olsen, K. (2007). NIST Atomic Spectra Database (version 3.1.2), [Online]. Available: http://physics.nist.gov/asd3 [2007, May 8]. National Institute of Standards and Technology, Gaithersburg, MD.

3.8. Symmetries

"I cannot believe that God is a weak left-hander…"
— Wolfgang Pauli

3.8.1. Spatial inversion

Spatial inversion (conventionally called the parity operation, *P*, though we will use the letter *M* for mirroring) is the process of inverting the three spatial axes. This operation corresponds to a mirror image followed by a 180 degree rotation about the axis perpendicular to the mirror. Since rotation does not affect any physical laws, we will sometimes substitute the term "mirror image" for "spatial inversion" when referring to general physical consequences. Parity conservation is generally taken to mean that when spatial inversion is applied to any physical process, the resulting process is equally frequent in nature. Parity violation means that a process and its mirror image are not equally likely, and maximal parity violation means that spatial inversion of a physical process yields a process with no physical interpretation.

In this chapter, we are not interested in the relative frequency of occurrence of events and their mirror images. We are only concerned with the question of maximal parity violation: "Is the mirror image process possible in nature or not?" We will refer to maximal parity violation as "mirror asymmetry", and existence of a mirror image process as "mirror symmetry."

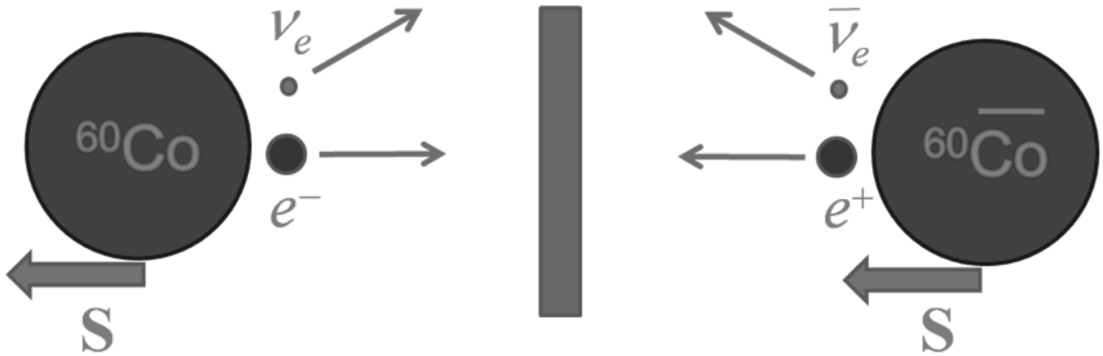

Figure 3.10 Beta decay of ^{60}Co (left) has a mirror image which is consistent with beta decay of its antimatter counterpart $^{60}\overline{\text{Co}}$ (right). The variable S represents the nuclear spin.

When viewed in a mirror, all known physical processes appear to proceed as if matter and anti-matter were exchanged. An example is the beta decay of ^{60}Cobalt shown in

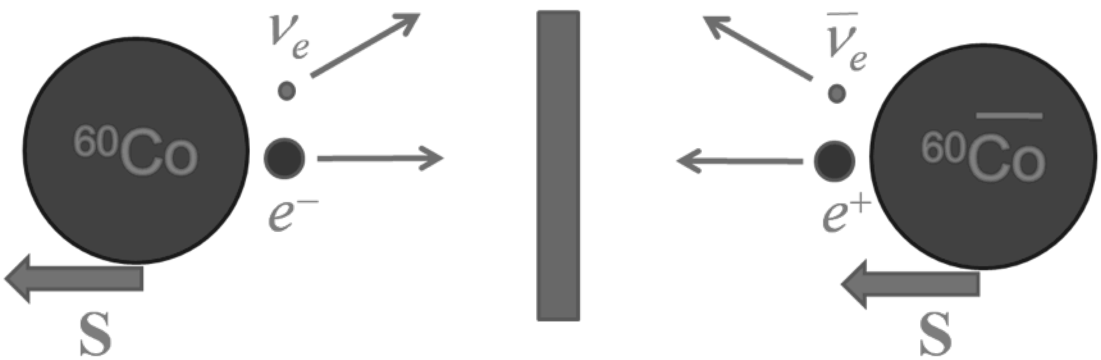

Figure 3.10. The simplest explanation for this observation is that spatial inversion exchanges matter and anti-matter. The mathematical basis for this explanation was derived by Close [2011b] as follows.

Let us consider how the wave function changes under spatial inversion.

Conventional parity operator

Dirac's original equation for a free particle has the form:

$$\partial_t \psi + c\beta_1 \sigma_i \partial_i \psi = -i\Omega\beta_3\psi \qquad \text{(3-222)}$$

where $\Omega \equiv mc^2/\hbar$. The β-matrices may be taken as:

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad \beta_2 = \begin{pmatrix} 0 & 0 & -\tilde{\mathrm{i}} & 0 \\ 0 & 0 & 0 & -\tilde{\mathrm{i}} \\ \tilde{\mathrm{i}} & 0 & 0 & 0 \\ 0 & \tilde{\mathrm{i}} & 0 & 0 \end{pmatrix}; \quad \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(3-223)

Where $\tilde{\mathrm{i}}$ is the pseudoscalar imaginary, as will be seen below.

The spin matrices σ^i utilize a true scalar imaginary ($\bar{\mathrm{i}}$):

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -\bar{\mathrm{i}} & 0 & 0 \\ \bar{\mathrm{i}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\mathrm{i}} \\ 0 & 0 & \bar{\mathrm{i}} & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(3-224)

Multiplying the Dirac equation by $\psi^\dagger$ and adding the Hermitian conjugate equation yields a continuity equation:

$$\partial_t\left[\psi^\dagger\psi\right] + \nabla\cdot\left[\psi^\dagger\beta_1\boldsymbol{\sigma}\psi\right] = 0 \tag{3-225}$$

This relationship is sufficient to establish the probability density ($\psi^\dagger\psi$) and current ($\psi^\dagger\beta_1\boldsymbol{\sigma}\psi$) as the components of a Lorentz four-vector.

Although the above analysis is satisfactory, it is currently fashionable to use the notation:

$$\gamma^0 \equiv \beta_3;\ \gamma^5 \equiv \beta_1;\ \gamma^i \equiv \gamma^0\gamma^5\sigma_i \tag{3-226}$$

and multiply each term in the original Dirac equation (3-222) by γ^0 to obtain:

$$\gamma^0\partial_t\psi + c\gamma^i\partial_i\psi \equiv \gamma^\mu\partial_\mu\psi = -\mathrm{i}\Omega\psi \tag{3-227}$$

This procedure cannot have any effect on the transformation properties of the Dirac matrices.

The conventional parity operator P is assumed to have the form: $P\psi(\mathbf{r}) = U\psi(-\mathbf{r})$. It is derived from the requirement that the Dirac equation in the form (3-227) be invariant with respect to the transformation:

$$\gamma^0\partial_t U\psi(-\mathbf{r}) + c\gamma^i\partial_i U\psi(-\mathbf{r}) + \mathrm{i}\Omega U\psi(-\mathbf{r}) = 0 \tag{3-228}$$

Inverting the parity operator yields:

$$U^{-1}\gamma^0 U\partial_t\psi(\mathbf{r}) - cU^{-1}\gamma^i U\partial_i\psi(\mathbf{r}) + \mathrm{i}\Omega U^{-1}U\psi(\mathbf{r}) = 0 \tag{3-229}$$

Equivalence with the original Dirac equation requires:

$$U^{-1}\gamma^0 U = \gamma^0$$
$$U^{-1}\gamma^i U = -\gamma^i \qquad (3\text{-}230)$$
$$U^{-1}U = 1$$

These conditions are satisfied by $U=\gamma^0$. Within an arbitrary phase factor the conventional parity operator is therefore:

$$P\psi(\mathbf{r}) = \gamma^0\psi(-\mathbf{r}) = \beta_3\psi(-\mathbf{r}) \qquad (3\text{-}231)$$

There are two problems with this derivation. First, the form $P\psi(\mathbf{r}) = U\psi(-\mathbf{r})$ is not the most general possible operator. For example, the conventional charge conjugation operator includes complex conjugation. Second, the matrix $\gamma^0 = \beta_3$ is not inverted because it is presumed to represent a temporal component of a four-vector. This illusion is maintained by rewriting the probability density and current components as $\overline{\psi}\gamma^0\psi$ and $\overline{\psi}\gamma^i\psi$, respectively, with $\overline{\psi} \equiv \psi^\dagger\gamma^0$. This change of notation does not change the fact, however, that the probability density is independent of γ^0. The matrix associated with the temporal part of the probability current 4-vector is the identity matrix, not γ^0. This is an important flaw in the conventional derivation of the parity operator.

Since the 4-vector ($\psi^\dagger\psi$,$\psi^\dagger\gamma^5\boldsymbol{\sigma}\psi$) is indeed Lorentz-invariant, there is absolutely no basis for the claim that γ^0 is a temporal component. On the contrary, we will show that γ^0 is geometrically related to wave velocity and may quite reasonably be inverted by spatial inversion. We will see that the resulting spatial inversion operator inverts all of the terms in the modified Dirac equation (3-227).

New spatial inversion operator

In discussing spatial inversion, it will be necessary to define two different unit imaginary numbers. As defined above, the product of spin matrices is a true scalar with respect to spatial inversion:

$$\bar{\mathrm{i}} \equiv \sigma^1\sigma^2\sigma^3 \qquad (3\text{-}232)$$

The σ-matrices are not involved in spatial inversion, which inverts the wave velocity but not the spin. However, we can identify three matrices associated with polar vectors which have the same algebra as the σ-matrices.

The β matrices define directions relative to the velocity vector $\langle c\beta_1\boldsymbol{\sigma}\rangle = \langle c\gamma^5\boldsymbol{\sigma}\rangle$, where the brackets indicate expectation value. One can also define absolute vectors $(\langle\beta_1\boldsymbol{\sigma}\rangle,\langle\beta_2\boldsymbol{\sigma}\rangle,\langle\beta_3\boldsymbol{\sigma}\rangle)$. If the wave function is an eigenfunction of velocity aligned with a spatial axis *xv* so that $c\beta_1\sigma_v\psi = c\psi$, then (using $\sigma_v^2 = 1$):

$$\psi^\dagger c\beta_3\sigma_v\psi = \left[\psi^\dagger\beta_1\sigma_v\right]c\beta_3\sigma_v\left[\beta_1\sigma_v\psi\right] = -\psi^\dagger c\beta_3\sigma_v\psi = 0$$
$$\psi^\dagger c\beta_2\sigma_v\psi = \left[\psi^\dagger\beta_1\sigma_v\right]c\beta_2\sigma_v\left[\beta_1\sigma_v\psi\right] = -\psi^\dagger c\beta_2\sigma_v\psi = 0 \tag{3-233}$$

These results follow from the fact that β_1 is a reflection operator for both β_2 and β_3, and the only number equal to its negative is zero. Therefore $\langle\beta_2\boldsymbol{\sigma}\rangle$ and $\langle\beta_3\boldsymbol{\sigma}\rangle$ are indeed perpendicular to velocity $\langle\beta_1\boldsymbol{\sigma}\rangle$ for velocity eigenfunctions. For example, in our notation the wave function $\psi_1 = (1 \quad 0 \quad 0 \quad 1)^T$ is a simultaneous eigenfunction of $\beta_1\sigma_1$, $\beta_2\sigma_2$, and $\beta_3\sigma_3$. Therefore the three vectors $\langle\beta_1\boldsymbol{\sigma}\rangle$, $\langle\beta_2\boldsymbol{\sigma}\rangle$, and $\langle\beta_3\boldsymbol{\sigma}\rangle$ are mutually orthogonal vectors (left-handed) in three dimensional space, at least for velocity eigenfunctions. The vector $\langle\beta_1\boldsymbol{\sigma}\rangle$ is parallel to $\hat{\mathbf{x}}_1$. Rotation of the vector $\langle\beta_1\boldsymbol{\sigma}\rangle$ by −90 degrees about the relative vector β_2 yields $\langle\beta_3\boldsymbol{\sigma}\rangle$, which is parallel to $\hat{\mathbf{x}}_3$. This is of course the same as rotation of $\hat{\mathbf{x}}_1$ by −90 degrees about $\hat{\mathbf{x}}_2$, which is associated with the matrix σ_2. It is therefore clear that for velocity eigenfunctions, the relative vectors represented by ($\beta_1, \beta_2, \beta_3$) are geometrically equivalent to the absolute vectors represented by ($\sigma_1, \sigma_2, \sigma_3$). We assume that all three vectors $\langle\beta_1\boldsymbol{\sigma}\rangle$, $\langle\beta_2\boldsymbol{\sigma}\rangle$, and $\langle\beta_3\boldsymbol{\sigma}\rangle$ are polar vectors so that the vector space ($\beta_1, \beta_2, \beta_3$) does not have mixed parity.

The matrix factor $\gamma^0 = \beta_3$ in the conventional parity operator represents a rotation by 180 degrees about the β_3 axis ($\hat{\mathbf{x}}_3$ in our example). This operation inverts only two of the three orthogonal vectors associated with velocity.

Compare this situation with classical transverse waves in a solid. We could define an operator (analogous to the Dirac *P* operator) which reflects the equilibrium position of each point in the solid, and also reflects the wave velocity direction. We also invert local displacements and velocities along one of the two axes perpendicular to the wave velocity. The resulting "reflected" wave would propagate along just as one would expect for the spatially inverted wave. But of course the operator we defined is not the spatial inversion operator, because we failed to invert one of the axes of the local displacement and velocity of the solid medium (in total we inverted two of the three local axes, corresponding to a 180° rotation about the third axis). Similarly, the Dirac *P* operator inverts the "wave" (or "particle") velocity direction, but inverts only one of two other quantities which are geometrically related to the "wave" velocity (a 180° rotation in the velocity-representation space). We will derive a new spatial inversion operator which inverts all three vectors $\langle\beta_1\boldsymbol{\sigma}\rangle$, $\langle\beta_2\boldsymbol{\sigma}\rangle$, and $\langle\beta_3\boldsymbol{\sigma}\rangle$ associated with velocity.

The spin matrices σi are components of a pseudovector and should not be inverted. Therefore the spatial inversion must be accomplished by inverting the three relative matrices ($\beta_1, \beta_2, \beta_3$). This requires that the associated imaginary $\tilde{\mathrm{i}}$ be a pseudoscalar, as assumed above. The unit imaginary associated with mass is assumed to be a pseudoscalar since it is multiplied by (β_3) in the original Dirac equation.

The roles of the different imaginaries can be clarified by factoring the Dirac wave function in a manner similar to that of Hestenes [1967]:

$$\psi(\mathbf{r}) \equiv a^{1/2} \exp(\bar{\mathrm{i}}\,\sigma_i \varphi_i) \exp(\beta_1 \sigma_i \alpha_i) \exp(\tilde{\mathrm{i}}\,\beta_1 \zeta)\psi_0 \tag{3-234}$$

It is clear that $\bar{\mathrm{i}}\,\sigma_i = \varepsilon_{ijk} \sigma_j \sigma_k / 2$ is associated with rotation in the plane orthogonal to the *xi* axis. Similarly, $\tilde{\mathrm{i}}\,\beta_1 = \beta_2 \beta_3$ is associated with rotation in the velocity-representation space.

Next we define a new wave function in which all imaginary pseudoscalar factors are inverted: $\psi^{\#}(\tilde{\mathrm{i}}) \equiv \psi(-\tilde{\mathrm{i}})$. This pseudoscalar conjugation operation differs from complex conjugation, which inverts both scalar and pseudoscalar imaginaries. Pseudoscalar conjugation inverts $\langle \beta_2 \rangle$ since:

$$\psi^{\#\dagger} \beta_2 \psi^{\#} = \left[\psi^{\dagger} \beta_2{}^{\#} \psi\right]^{\#} = \left[\psi^{\dagger} [-\beta_2] \psi\right]^{\#} = \psi^{\dagger} [-\beta_2] \psi \tag{3-235}$$

The spatial inversion (or mirroring operator *M*) which inverts all of the relative velocity vectors, is then (within an arbitrary phase factor):

$$M\psi(\mathbf{r}) \equiv \psi_M(\mathbf{r}) = \beta_2 \psi^{\#}(-\mathbf{r}) \tag{3-236}$$

This operator inverts observables computed from β_1, β_2, and β_3 independently of the change in sign of $\mathbf{r}$.

The Dirac equation for a particle in electromagnetic potentials is:

$$\left[\partial_t + c\beta_1 \sigma^i \partial_i + \tilde{\mathrm{i}}\,\beta_3 \Omega + \mathrm{i} e \Phi - \mathrm{i} e \beta_1 \sigma^i \mathbf{A}_i\right] \psi = 0 \tag{3-237}$$

When applied to this equation, the parity operator inverts β_3, β_1, $\tilde{\mathrm{i}}$, and ∂_i (the matrices are inverted because they anti-commute with β_2). Denoting spatially inverted quantities with subscript *M*, the spatially inverted Dirac equation is:

$$\left[\partial_t + c\beta_1 \sigma^i \partial_i + \tilde{\mathrm{i}}\,\beta_3 \Omega_M + \mathrm{i}^{\#} e_M \Phi_M + \mathrm{i}^{\#} \beta_1 \sigma^i (e_M \mathbf{A}_{Mi})\right] \psi_M = 0 \tag{3-238}$$

We assume $\mathbf{\Omega}_M = \mathbf{\Omega}$. The transformed equation has the same form as the original Dirac equation except for the sign of the vector potential term. This sign change is necessary for consistency with gauge transformations. The gauge transformation

$$\begin{aligned} e\Phi' &= e\Phi + \partial_t \chi \\ eA_i' &= eA_i - \nabla \chi \\ \psi' &= \exp(-\mathrm{i}\,\chi) \end{aligned} \tag{3-239}$$

suggests that the scalar potential may be regarded as a time derivative and the vector potential may be regarded as a spatial derivative. Taking $\mathbf{\Phi} \equiv \partial_t g$ and $\mathbf{A} \equiv \nabla \times \mathbf{G} + \nabla g$ would leave the form of the equation invariant:

$$\left[\partial_t + c\beta_1 \sigma^i \partial_i + \tilde{\mathrm{i}}\,\beta_3 \Omega_M + \mathrm{i}^{\#} e_M \partial_t g_M - \mathrm{i}^{\#} e_M \beta_1 \boldsymbol{\sigma} \cdot (\nabla \times \mathbf{G}_M + \nabla g_M)\right] \psi_M = 0 \tag{3-240}$$

The scalar and vector potentials must have opposite spatial inversion eigenvalues. We will assume that:

$$M\left[\mathrm{i}e\Phi(\mathbf{r})\right] = \mathrm{i}^{\#}e_M\Phi_M = -\mathrm{i}e\Phi(-\mathbf{r})$$
$$M\left[\mathrm{i}e\mathbf{A}(\mathbf{r})\right] = \mathrm{i}^{\#}e_M\mathbf{A}_M = \mathrm{i}e\mathbf{A}(-\mathbf{r}) \tag{3-241}$$

The transformed Dirac equation is then:

$$\left[\partial_t + c\beta_1\sigma^i\partial_i + \tilde{\mathrm{i}}\beta_3\Omega_M - \mathrm{i}e_M\Phi(-\mathbf{r}) - \mathrm{i}e_M\beta_1\sigma_i A_i(-\mathbf{r})\right]\psi_M = 0$$

With these transformation properties, we will show that the new parity operator is consistent with an exchange of matter and anti-matter.

Eigenfunctions and eigenvalues

Next we consider the effect of the new parity operator on the eigenvalue equation. For simplicity we assume the vector potential $\mathbf{A}$ to be zero. Assuming temporal dependence $\exp(-\mathrm{i}\,Et)$, the eigenvalue equation is:

$$\left[-\mathrm{i}\,E + \mathrm{i}\,e\Phi + c\beta_1\boldsymbol{\sigma}\cdot\nabla\right]\psi = -\tilde{\mathrm{i}}\,\Omega\beta_3\psi \tag{3-242}$$

The operator $\boldsymbol{\sigma}\cdot\nabla\psi$ can be factored:

$$\boldsymbol{\sigma}\cdot\nabla\psi = \sigma_r\left[\partial_r + \mathrm{i}\frac{\boldsymbol{\sigma}}{r}\cdot\left[\mathbf{r}\times\nabla\right]\right]\psi = \sigma_r\left[\partial_r - \frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right]\psi \tag{3-243}$$

The two-component angular solutions of the eigenvalue equations $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(+)}_{l,m} = l = -1+\kappa$ and $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(-)}_{l,m} = -\left[l+2\right] = -1-\kappa$ are well known [Bjorken and Drell 1964]. These two angular solutions are related by $\sigma_r\Phi^{(+)}_{l,m} = \Phi^{(-)}_{l,m}$ and yield opposite eigenvalues under coordinate inversion ($\mathbf{r}\to\mathbf{r}$). Only the true scalar imaginary $\bar{\mathrm{i}}$ can appear within these functions.

Denote two wave functions as:

$$\psi^{(+)} = \frac{1}{r}\begin{bmatrix}\tilde{\mathrm{i}}\,G\Phi^{(+)}_{l,m}\\ F\Phi^{(-)}_{l,m}\end{bmatrix} \quad \text{or} \quad \psi^{(-)} = \frac{1}{r}\begin{bmatrix}\tilde{\mathrm{i}}\,F\Phi^{(-)}_{l,m}\\ G\Phi^{(+)}_{l,m}\end{bmatrix} \tag{3-244}$$

Each of these is an eigenfunction of the conventional parity operator, but they are exchanged by the new spatial inversion operator:

$$M\psi^{(+)}(\mathbf{r}) = \gamma^4\psi^{(+)\#}(-\mathbf{r}) = (-)^l\psi^{(-)}(\mathbf{r})$$
$$M\psi^{(-)}(\mathbf{r}) = \gamma^4\psi^{(-)\#}(-\mathbf{r}) = (-)^{l+1}\psi^{(+)}(\mathbf{r}) \tag{3-245}$$

Using $\psi^{(+)}$ in the (original) Dirac equation yields the coupled radial equations:

$$[E - e\Phi - \Omega]G + c\left[\partial_r + \frac{\kappa}{r}\right]F = 0$$

$$[E - e\Phi + \Omega]F - c\left[\partial_r - \frac{\kappa}{r}\right]G = 0 \tag{3-246}$$

$\psi^{(-)}$ yields similar coupled equations with opposite sign of E and $e\Phi$, as expected for exchange of matter and anti-matter (one interpretation is that antiparticles represent positive-energy "holes" in a sea of negative-energy particles. If we want two positive energy solutions related by spatial inversion then we need to use different eigenfunctions). The energy levels for $\psi^{(+)}$ in a negative Coulomb potential are therefore equal and opposite to the energy levels of $\psi^{(-)}$ in a positive Coulomb potential. The need for this result was the reason for assuming that the parity operator locally inverts the scalar potential term $\mathrm{i}e\Phi(\mathbf{r})$.

Weak interactions

The projection operator for left-handed spinor components is:

$$\psi_L = (I - \beta_1)\psi \tag{3-247}$$

The unit matrix I is a scalar and β_1 is a pseudoscalar. However, the projection operator does not violate mirror symmetry so long as the reflected counterpart $\psi_R = (I + \beta_1)M\psi$ is as physically plausible as the original projected wave function. Since the new spatial inversion operator exchanges matter and antimatter, all of the elementary particles involved in the weak interaction do in fact have spatially reflected counterparts in nature (electrons and positrons, left-handed neutrinos and right-handed anti-neutrinos, etc.). The mathematical form of the weak vertex factor is entirely consistent with mirror symmetry.

Comparison with conventional *PC*

The conventional *PC* operator is:

$$PC\psi(\mathbf{r}) = \mathrm{i}\gamma^0\gamma^2\psi^*(-\mathbf{r}) = \mathrm{i}\gamma^5\sigma_2\psi^*(-\mathbf{r}) \tag{3-248}$$

This differs from our spatial inversion operator by an arbitrary phase factor, the factor of $\gamma^0\sigma_2$ and conjugation of the scalar imaginary (denoted by $\psi \to \psi^{*\#}$). The factor σ_2 is, within a phase factor, simply a rotation by π about the $x2$ axis: $\sigma_2 = \bar{\mathrm{i}}\exp(-\bar{\mathrm{i}}\sigma_2\pi/2)$. Complex conjugation of the scalar imaginary inverts the spin component $S2$:

$$PCS_2 = \psi^{*\dagger}\sigma_2\psi^* = \left[\psi^\dagger\sigma_2{}^*\psi\right]^* = \left[\psi^\dagger[-\sigma_2]\psi\right]^* = -\psi^\dagger\sigma_2\psi = -S_2 \tag{3-249}$$

Therefore the net effect of $\psi \to \sigma^2\psi^{*\#}$ is to invert the spin.

The additional factor of γ^0 inverts velocity by rotation of the velocity-representation matrices. Applied to the matter and anti-matter eigenfunctions, it is equivalent to inverting the spatial arguments in the wave functions. Therefore the conventional *PC* operator, though it exchanges matter and anti-matter, differs significantly from the new spatial inversion operator *M*.

3.8.2. Time reversal

Physically, time reversal must invert the time derivative operator, velocity, and spin independently of the change in argument. One of the electromagnetic potentials must also be inverted. Velocity and spin are both inverted by the transformation:

$$B\psi(t) = \psi_B(t) = \sigma^2 \psi^{*\#}(-t) \tag{3-250}$$

The velocity-representation space $(\gamma^5, \gamma^4, \gamma^0)$ is unaffected by this transformation. By contrast, the conventional time reversal operator $T\psi(t) = \mathrm{i}\,\sigma^2 \psi^*(-t)$ inverts γ^4 but not other matrices of velocity-representation space. This suggests that the conventional time reversal operator is also incorrect. However, unlike the conventional parity transformation, there is no empirical evidence to validate this claim.

Applied to the Dirac equation, the new time reversal operator yields:

$$\begin{aligned} &B\left\{\left[\partial_t + c\gamma^5\sigma^i\partial_i + \tilde{\mathrm{i}}\,\gamma^0\Omega + \mathrm{i}e\Phi - \mathrm{i}e\gamma^5\sigma^i A_i\right]\psi\right\} \\ &= -\left[\partial_t + c\gamma^5\sigma^i\partial_i - \tilde{\mathrm{i}}\,\gamma^0\Omega_B - \mathrm{i}^{*\#}e_B\Phi_B - \mathrm{i}^{*\#}\gamma^5\sigma^i e_B A_{Bi}\right]\psi_B = 0 \end{aligned} \tag{3-251}$$

We recover the original form of the Dirac equation if $\Omega_B = -\Omega$ (i.e. Ω is an eigenvalue of an operator which transforms like a time derivative) and the potentials are interpreted as derivatives.

We assume the potentials to transform as:

$$\begin{aligned} &B[\mathrm{i}e\Phi(t)] = \mathrm{i}^{*\#}e_B\Phi_B = \mathrm{i}e\Phi(-t) \\ &B[\mathrm{i}e\mathbf{A}] = \mathrm{i}^{*\#}e_B\mathbf{A}_B = -\mathrm{i}e\mathbf{A}(-t) \end{aligned} \tag{3-252}$$

According to our interpretation of matter and anti-matter as mirror-images, time reversal does not exchange the two.

3.8.3. Combined Transformations

The combined *MB* transformation is:

$$MB\psi(\mathbf{r},t) = \gamma^4\sigma_2\psi^*(-\mathbf{r},-t) = \mathrm{i}\,\gamma^2\psi^*(-\mathbf{r},-t) \tag{3-253}$$

This is closely related to the conventional charge conjugation transformation C:

$$C\psi(\mathbf{r},t) = \gamma^4\sigma_2\psi^*(\mathbf{r},t) = \gamma^2\psi^*(\mathbf{r},t) \tag{3-254}$$

The conventional charge conjugation operator inverts the spin and velocity in place, without inverting the spatial or temporal coordinates. In terms of dynamical behavior, charge conjugation has the same effect as inverting the sign of the electromagnetic potentials in the Dirac equation.

The conventional *PT* transformation is:

$$PT\psi(\mathbf{r},t) = \gamma^0 \sigma_2 \psi^*(-\mathbf{r},-t) \tag{3-255}$$

This differs from the new *MB* transformation by the factor γ^5, which rotates the velocity-representation space by 180 degrees.

The conventional *PCT* transformation is:

$$PCT\psi(\mathbf{r},t) = \gamma^5 \psi(-\mathbf{r},-t) \tag{3-256}$$

This transformation is the conventional theoretical relation between matter and antimatter. Compared with the *MB* operator, it differs only by charge conjugation (which has similar effect to restoring the potentials inverted by *MB*) and by the factor γ^5.

3.9. Mathematical and Physical Properties of Spinors

"*…our present thinking about quantum mechanics is infested with the deepest misconceptions.*"
—Stephen Gull, Anthony Lasenby, and Chris Doran [1993]

3.9.1. Spinors and Inner Products

An understanding of some mathematical properties of spinors will be useful. Expressions for physical quantities (e.g. Q) are computed from operators (e.g. *Q*) in the form:

$$\mathrm{Q} = \frac{1}{2}\left[\left[Q\psi\right]^\dagger \psi + \psi^\dagger\left[Q\psi\right]\right] = \frac{1}{2}\left[\psi^\dagger\left[Q\psi\right] + \left[\psi^\dagger\left[Q\psi\right]\right]^\dagger\right] \tag{3-257}$$

Since the adjoint of a scalar is its complex conjugate, the physical quantity Q is real-valued. When integrated over space, such expressions take the form of an inner product:

$$\langle \mathrm{Q} \rangle = \frac{1}{2}\left[(f,g) + (g,f)\right] = \frac{1}{2}\int\left[f^\dagger g + g^\dagger f\right] d^3r \tag{3-258}$$

The quantity <Q> is the integrated value (or expectation value in QM).

A complete space of functions with an inner product satisfying some simple properties (*e.g.* linearity) is called a 'Hilbert space'. It suffices for our purposes to say that the inner product defined above satisfies all of the necessary criteria.
[Note: the inner product is often defined using only one of the terms in the integrand above (without the factor of one-half). With this definition local densities may be complex even though the integral is real.]

The inner product between two spinor functions is analogous to the dot product between two vectors or the correlation between two scalar functions. The inner product of a spinor function with itself is its positive-definite magnitude:

$$\|f\|^2 = (f,f) = \int f^\dagger f \, d^3r \geq 0 \tag{3-259}$$

In terms of components this is:

$$\left(f,f\right)=\int\sum_{\alpha} f_{\alpha}^{*}f_{\alpha}\, d^{3}r \geq 0 \tag{3-260}$$

The local projection $p_{\Phi}\psi(\mathbf{r})$ of one function $\psi(\mathbf{r})$ onto another function $\Phi(\mathbf{r})$ is defined as:

$$p_{\Phi}\psi(\mathbf{r})=\frac{\Phi^{\dagger}(\mathbf{r})\psi(\mathbf{r})}{\Phi^{\dagger}(\mathbf{r})\Phi(\mathbf{r})}\Phi(\mathbf{r}) \tag{3-261}$$

The global projection $P_{\Phi}\psi(\mathbf{r})$ of one function ψ onto another function Φ is defined as:

$$P_{\Phi}\psi(\mathbf{r})=\frac{(\Phi,\psi)}{\|\Phi\|^{2}}\Phi(\mathbf{r}) \tag{3-262}$$

The term 'projection' by itself generally refers to the global projection in the literature. For comparison, the projection of a vector **a** onto a vector **b** is the component of **a** that is parallel with **b**:

$$P_{\mathbf{b}}\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^{2}}\mathbf{b}=\left[\mathbf{a}\cdot\hat{\mathbf{b}}\right]\hat{\mathbf{b}} \tag{3-263}$$

If an operator has Hermitian ($H^{\dagger}=H$) and anti-Hermitian ($A^{\dagger}=-A$) parts, then only the Hermitian part contributes to the physical value:

$$\mathrm{Q}=\frac{1}{2}\left[\left[\left[H+A\right]\psi\right]^{\dagger}\psi+\psi^{\dagger}\left[\left[H+A\right]\psi\right]\right]=\frac{1}{2}\left[\psi^{\dagger}\left[H^{\dagger}+A^{\dagger}+H+A\right]\psi\right]=\psi^{\dagger}H\psi \tag{3-264}$$

From this we can conclude that the condition for a real-valued inner product is that the operator is Hermitian ($Q^{\dagger}=Q$). For example consider the spatial derivative $\nabla_{j}=\partial/\partial x_{j}$:

$$\left(f,\nabla_{j}f\right)=\int f^{\dagger}\nabla_{j}f\, d^{3}r \tag{3-265}$$

The adjoint is:

$$\left(f,\nabla_{j}f\right)^{\dagger}=\int\left[\nabla_{j}f\right]^{\dagger}f\, d^{3}r \tag{3-266}$$

Integration by parts yields:

$$\int\left[\nabla_{j}f\right]^{\dagger}f\, d^{3}r=\int\nabla_{j}f^{\dagger}f\, d^{3}r=\left[\iint dS_{j}\, f^{\dagger}f\right]_{x_{1j}}^{x_{2j}}-\iiint d^{3}r\, f^{\dagger}\left[\nabla_{j}f\right] \tag{3-267}$$

We assume that the spinor functions fall to zero prior to reaching the boundary of integration (*i.e.* that the boundary is sufficiently far that there is no contribution to the volume integral outside the boundary). This assumption allows us to discard the boundary term, but limits our ability to give physical interpretation to the local functions. Assuming the boundary contribution to be zero, we have:

$$\left(f,\nabla_j f\right)^\dagger = -\left(f,\nabla_j f\right) \tag{3-268}$$

Hence the spatial derivative is an anti-Hermitian operator (minus sign rather than plus sign).

Clearly this property holds for all components of the gradient, so we can write:

$$\left(f,\nabla f\right)^\dagger = -\left(f,\nabla f\right) \tag{3-269}$$

Which leads to the rather obvious expression for the integrated value:

$$\langle\nabla\rangle = \int\left[\nabla f\right]^\dagger f + f^\dagger \nabla f\, d^3r = \int - f^\dagger \nabla f\, d^3r + f^\dagger \nabla f\, d^3r = 0$$

This relationship in operator form is:

$$\left[\nabla f\right]^\dagger = \left[\nabla^\dagger f\right]^\dagger = -f^\dagger \nabla \tag{3-270}$$

Note that the form of the gradient operator is not changed by the adjoint operation ($\nabla^\dagger = \nabla$). The sign change comes from transposing the operator from the left to the right side (via integration by parts). Note that:

$$\nabla\left[f^\dagger f\right] = \left[\nabla f^\dagger\right] f + f^\dagger\left[\nabla f\right] \tag{3-271}$$

This expression is obviously not zero in general, but its volume integral is zero as long as the function f falls off sufficiently rapidly near the integration boundaries.

It is simple to construct a Hermitian operator from the gradient operator by multiplying it with the unit imaginary:

$$\left(f,\mathrm{i}\nabla f\right)^\dagger = +\left(f,\mathrm{i}\nabla f\right) \tag{3-272}$$

Matrix Algebra

Before proceeding further, it will be useful to tabulate some relationships between matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3-273}$$

In the Dirac representation of quantum mechanics these matrices represent $\left(\gamma^5, \mathrm{i}\gamma^5\gamma^0, \gamma^0\right)$, respectively.

In spherical coordinates the sigma matrices are:

$$\sigma_r = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} & 0 & 0 \\ \sin\theta\, e^{i\phi} & -\cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta\, e^{-i\phi} \\ 0 & 0 & \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\sigma_\theta = \begin{pmatrix} -\sin\theta & \cos\theta\, e^{-i\phi} & 0 & 0 \\ \cos\theta\, e^{i\phi} & \sin\theta & 0 & 0 \\ 0 & 0 & -\sin\theta & \cos\theta\, e^{-i\phi} \\ 0 & 0 & \cos\theta\, e^{i\phi} & \sin\theta \end{pmatrix}$$

$$\sigma_\phi = \begin{pmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{pmatrix} \tag{3-274}$$

The operator $\boldsymbol{\sigma}\cdot\nabla\psi$ therefore yields:

$$\boldsymbol{\sigma}\cdot\nabla\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix} \cos\theta[\partial\psi_1/\partial r] + \sin\theta\, e^{-i\phi}\,[\partial\psi_2/\partial r] \\ \sin\theta\, e^{i\phi}\,[\partial\psi_1/\partial r] - \cos\theta[\partial\psi_2/\partial r] \\ \cos\theta[\partial\psi_3/\partial r] + \sin\theta\, e^{-i\phi}\,[\partial\psi_4/\partial r] \\ \sin\theta\, e^{i\phi}\,[\partial\psi_3/\partial r] - \cos\theta[\partial\psi_4/\partial r] \end{pmatrix}$$
$$+\frac{1}{r}\begin{pmatrix} -\sin\theta[\partial\psi_1/\partial\theta] + \cos\theta\, e^{-i\phi}[\partial\psi_2/\partial\theta] \\ \cos\theta\, e^{i\phi}\,[\partial\psi_1/\partial\theta] + \sin\theta[\partial\psi_2/\partial\theta] \\ -\sin\theta[\partial\psi_3/\partial\theta] + \cos\theta\, e^{-i\phi}[\partial\psi_4/\partial\theta] \\ \cos\theta\, e^{i\phi}\,[\partial\psi_3/\partial\theta] + \sin\theta[\partial\psi_4/\partial\theta] \end{pmatrix} + \frac{1}{r\sin\theta}\begin{pmatrix} -ie^{-i\phi}\,[\partial\psi_2/\partial\phi] \\ ie^{i\phi}\,[\partial\psi_1/\partial\phi] \\ -ie^{-i\phi}\,[\partial\psi_4/\partial\phi] \\ ie^{i\phi}\,[\partial\psi_3/\partial\phi] \end{pmatrix} \tag{3-275}$$

In cylindrical coordinates ($r_\perp, \phi, z$) the matrices are:

$$\sigma_{r_\perp} = \sigma_x \cos\phi + \sigma_y \sin\phi = \begin{pmatrix} 0 & e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_\phi = -\sigma_x \sin\phi + \sigma_y \cos\phi = \begin{pmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (3\text{-}276)$$

3.9.2. Wave Properties of Matter

We have shown that classical wave theory can describe Fermion dynamics. This result lends support to recent efforts to revive the classical aether (or ether) as a medium of propagation of matter waves. Duffy (2006) has surveyed modern aether theory.

The model of vacuum as an ideal elastic solid was quite successful in explaining classical properties of light in the 19th century (see e.g. Whittaker (1951)). Quantum effects are only apparent in interactions with matter, which might be interpretable as classical soliton waves. At present there appears to be no satisfactory description of rotational waves in an ideal elastic medium. Kleinert (1989) attempted to include rotations in the elastic energy but was compelled to introduce new elastic constants dependent on an arbitrary scale length. Close (2002) showed that torsion waves (with rotation axis parallel to wave velocity) can be described by a Dirac equation. In this book we use a wave equation with convection terms as the classical basis for the quantum mechanical momentum and spin operators. Schmelzer (2009) recently demonstrated that a cellular lattice model can yield the same group structure as the Standard Model. This model is astonishingly similar to the rotating elastic cell model which Maxwell used to derive the equations of electromagnetism (though the rolling particles bordering Maxwell's cells were replaced by an unspecified material between the lattice cells).

Many physical properties of matter can be derived from a wave model of matter. The Uncertainty Principle applies to all classical waves and represents a basic property of Fourier transformations. Lorentz invariance is also a property of waves, and Special Relativity is therefore a consequence of any wave theory of matter. For example, the relativistic phenomenon of time dilation is simply explained by the fact that stationary soliton waves execute periodic orbits (e.g. circles) whereas moving solitons execute orbits which have longer wave paths in each cycle (e.g. spiral or cycloidal). Hence a moving clock which counts soliton wave orbits ticks faster than a similar moving clock. Absolute motion with respect to the aether would not be detectable because without prior knowledge of absolute motion it is unknown whether a signal is Doppler shifted at the source or the receiver, or both.

There has been considerable interest in describing elementary particles as soliton (or particle-like) wave solutions of a nonlinear Dirac equation. See Rañada (1983) for a short review. More recent works include Fushchych and Zhdanov (1997), Gu (1998), Bohun and Cooperstock (1999), and Maccari (2006). These efforts all suffer from arbitrariness in the choice of nonlinearity. Identification of the Dirac equation with a second-order classical wave equation provides a simple means for interpreting, literally or analogously, any non-linear terms.

The Klein-Gordon (or relativistic Schrödinger) operator can be factored into a product of two Dirac operators acting on the wave polarization (or amplitude) **a**:

$$\left\{\partial_t^2 - c^2\nabla^2 + M^2\right\}\mathbf{a} = \left\{\gamma_0\partial_t + \gamma_i\partial_i + \mathrm{i}M\right\}\left\{\gamma_0\partial_t + \gamma_i\partial_i - \mathrm{i}M\right\}\mathbf{a} \tag{3-277}$$

where the commutation relations are:

$$\begin{aligned} \gamma_0{}^2 &= 1 \\ \gamma_0\gamma_i &= -\gamma_i\gamma_0 \\ \gamma_i\gamma_j &= -\delta_{ij} - \mathrm{i}\,\varepsilon_{ijk}\gamma_k \\ \mathrm{i}^2 &= -1 \end{aligned} \tag{3-278}$$

The quantities γ_μ and unit imaginary (i) have traditionally been regarded as matrices, but they can also be interpreted geometrically using multivariate vectors [Hestenes 1967, 1973, 1990]. The wave polarization **a** is a classical 3-vector in Galilean space-time. The Minkowski metric of relativity is introduced through the operators.

If we define a wave function:

$$\Psi \equiv \left\{\gamma_0\partial_t + \gamma_i\partial_i - \mathrm{i}M\right\}\mathbf{a} \tag{3-279}$$

then the resultant first-order Dirac equation is equivalent to the original Klein-Gordon equation:

$$\left\{\gamma_0\partial_t + \gamma_i\partial_i + \mathrm{i}M\right\}\Psi = 0 \tag{3-280}$$

In the above case the two Dirac operators have different sign for the mass term. Rowlands [1998, 2005, 2006] and Rowlands and Cullerne [2000] used a combination of multivariate 4-vectors and quaternions to write the Dirac equation in a nilpotent form in which the two successive Dirac operations are identical. This formulation yields an elegant classification of particle states within the Standard Model.

Standard solutions of the Klein-Gordon equation yield different energy eigenvalues than the Dirac equation (see e.g. Schiff [1968]). This result is quite peculiar given the fact that each component of the Dirac wave function actually satisfies the Klein-Gordon equation! Factoring the Klein-Gordon equation cannot change its eigenvalues. The problem is that in the usual analysis of Klein-Gordon, the angular functions are chosen to be eigenfunctions of the squared orbital angular momentum $L2$, whereas in the analysis of the Dirac equation the angular functions are eigenvalues of the squared total angular momentum $J\,2$. The difference is not in the equations, but in the choice of angular eigenfunctions. The usual analysis of the Klein-Gordon equation neglects the spin contribution from rotation of wave velocity. These solutions represent bosons with zero spin. Solutions obtained by using angular eigenvalues obtained from Dirac theory represent fermions with spin one-half.

In the next chapter we shall see that a scalar gravitational field and its effect on the space-time metric may be interpreted as a spatially varying light speed. See Whittaker (1954) for the historical development of this idea which originates with Einstein (1911, 1912) and has also been investigated more recently (de Felice (1971), Evans et al (2001)). This interpretation is consistent with general relativity, which also predicts a variation of light speed proportional to the gravitational potential (Einstein 1956). In an elastic solid aether, compression or variations in elasticity imply variable wave speed and hence provide a reasonable physical model for basic gravitational effects.

3.10. Summary

Even if you are a minority of one, the truth is the truth.
—Mohandas Gandhi

In this chapter we interpret the Dirac equation as a classical second-order wave equation for rotational waves in an elastic medium. The first order spatial and temporal derivatives are represented by a bispinor wave function. Half-integer spin is attributable to the co-existence of waves traveling in opposite directions along the gradient axis. The wave function can be factored into constant matrix, a single amplitude, a three-dimensional Lorentz velocity boost, rotation, and an arbitrary change of representation. Wave interference yields both the Pauli exclusion principle and the Lorentz force. The electromagnetic potentials represent wave interference. Interpreting the classical bispinor equation as describing an electron, it is found that the mass is associated with radially inward acceleration of the wave, suggestive of a soliton. The classical theory is consistent with parity conservation. Hence it appears that classical wave theory constitutes an intelligible basis for the physical attributes of matter.

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Figures

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Figure 3.1 .J. Thomson (1856-1940).
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Figure 3.2 Ernest Rutherford (1871-1937).

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Figure 3.3 Arnold J.W. Sommerfeld (1868-1951).

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Figure 3.4 Wolfgang Pauli (1900-1958).

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Figure 3.5 Werner Heisenberg (1901-1976).

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Figure 3.6 Erwin Schrodinger (1887-1961).

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Figure 3.7 Paul Dirac (1902-1984).

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Chapter 4. Wave Refraction and Gravity

"It is only the relation of the magnitude to the instrument that we measure, and if this relation is altered, we have no means of knowing whether it is the magnitude or the instrument that has changed."
–Henri Poincaré, *Science et Méthode* 1897

4.1. Introduction

"*Gravity is probably due to a change in the structure of the aether, produced by the presence of matter."*
– George Francis FitzGerald 1894

Isaac Newton [**Figure 4.1**] published his theory of gravity in *Principia* in 1687. Newton realized that a force proportional to the inverse square of the distance between two masses would yield elliptical planetary orbits with the sun at one focus of the ellipse. He conjectured that the gravitational force might represent a tendency of matter to move from denser to rarer regions of the aether. Tests of Newton's theory were sometimes difficult and required planetary observational data accumulated over long periods of time. For example, in 1784 Pierre-Simon Laplace [**Figure 4.2**] determined that the apparently secular (non-periodic) motions of Jupiter and Saturn were actually periodic with a period of 929 years, the frequency corresponding to the difference between five periods of Saturn and two periods of Jupiter. Although Newton's law eventually succeeded in explaining most astronomical observations, a few observations resisted interpretation. This included the rate of rotation of the elliptical axes of Mercury.

Figure 4.1 Isaac Newton (1643 – 1727)

Figure 4.2 Pierre-Simon Laplace (1749-1827)

Lóránd (or Roland) Eötvös [1891] [**Figure 4.3**] reported experimental results indicating that inertial mass and gravitational mass are exactly equal. Albert Einstein [1907] [**Figure 4.4**] then proposed the Principle of Equivalence between an accelerating reference frame and a gravitational field. He also deduced that the speed of light must vary in a gravitational field [Einstein 1911, 1912].

Figure 4.3 Lóránd Baron von Eötvös (1848-1919)

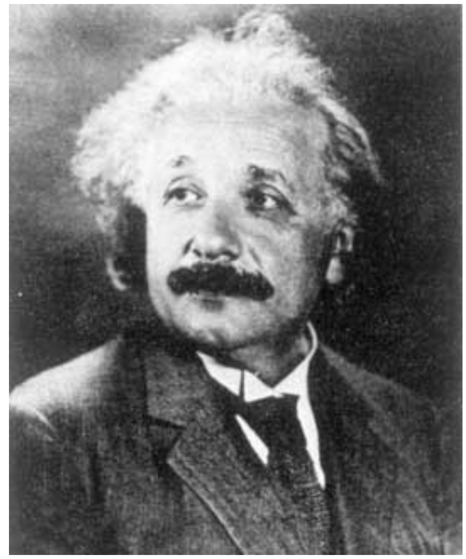

Figure 4.4 Albert Einstein (1879 – 1955)

Harry Bateman [1909] observed that the condition for propagation of light:

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \tag{4-1}$$

does not hold in a gravitational field. Instead a condition of the form:

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu = 0 \tag{4-2}$$

describes the propagation of light in a gravitational field which is characterized by the coefficients $g_{\mu\nu}$. Time is denoted by x_0 and the coefficient g_{00} is equal and opposite to the spatial coefficients g_{ii} in the absence of gravity.

Albert Einstein and Marcel Grossmann [1913] proposed that particle motion in a gravitational field is described as a geodesic in space-time determined by the variational equation:

$$\delta \int ds = 0 \tag{4-3}$$

with *ds* defined as above. Combined with an equation relating the metric coefficients with the energy tensor of matter, this idea formed the General Theory of Relativity [Einstein 1915a,b,c]. David Hilbert [1915] [**Figure 4.5**] showed that the entire theory could be formulated using a variational principle. Karl Schwarzschild [1916] [**Figure 4.6**] found exact solutions for a point mass.

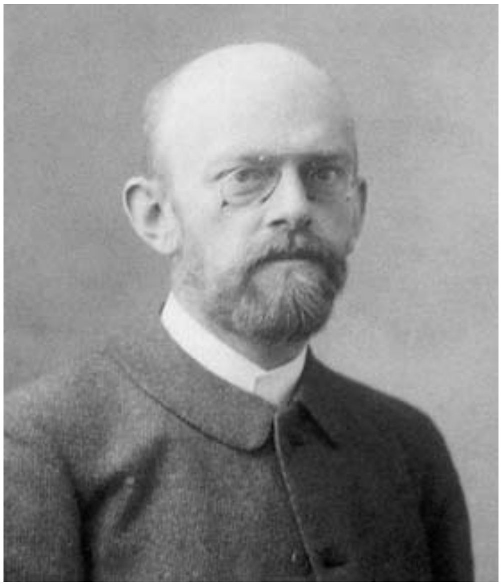

Figure 4.5 David Hilbert (1862 – 1943)

Figure 4.6 Karl Schwarzschild (1873-1916)

Many predictions of the General Theory have been successfully validated by experimental observations. In addition to the usual attraction between massive objects, the theory also accurately predicts deflection of light rays around massive objects, deviations from simple elliptical planetary orbits, and non-Euclidean curvature of space. The theory also predicts the existence of black holes: regions where gravity is so strong that light cannot escape. There is now very strong astronomical evidence of black holes, including one at the center of our own galaxy.

In this chapter we will compare wave refraction with General Relativity. In particular, we will use the analogy of compression of a wave-carrying medium, such as an elastic solid. Although Albert Einstein is sometimes credited with eliminating the need for an aether to carry light waves, his own view was that "…according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether" [Albert Einstein, 1920 Leiden Lecture]. But Einstein did not believe that the aether was a substance whose motions could be tracked. Dirac, however, concluded that "It is necessary to set up an action principle and to get a Hamiltonian formulation of the equations suitable for quantization purposes, and for this the aether velocity is required." [Dirac 1952].

Other investigators have attempted to model the vacuum as an elastic solid. Two recent efforts are those of Hatch [1992] and Karlsen [1998]. Gravity has been interpreted by many as refraction due to a variable index of refraction of space [Alsing et al. 2001, Anonymous 2002, Colsman 1997, Evans et al. 2001, de Felice 1971, Peters 1974]. Although many physicists believe that gravity should have a quantum mechanical description, the classical description adequately explains a wide range of gravitational phenomena.

4.2. Wave propagation in a non-uniform medium

> "*It is worth noting that, strictly speaking, there cannot be any point particles in general relativity. They have to be much larger than their Schwarzchild radius ...*"
> – Hagen Kleinert [1989]

Since elastic waves yield bispinor equations similar to the equations of quantum mechanics, it is natural to question whether elastic waves can produce gravity. A simple mechanism is that twisting of the medium can generate tension which causes the medium to compress. This effect can be easily observed using a rubber band. Twisting the rubber band stretches it, thereby generating tension which pulls inward from the ends. The square of the wave speed is inversely proportional to density and therefore decreases as one approaches the region of increased density. Since waves refract in the direction of decreased wave speed there is a mutual attraction between rotational waves [**Figure 4.7**]. This mechanism is consistent with the weakness of gravity with respect to other forces (it is a second-order effect), and also provides an explanation for gravity being an attractive rather than repulsive force.

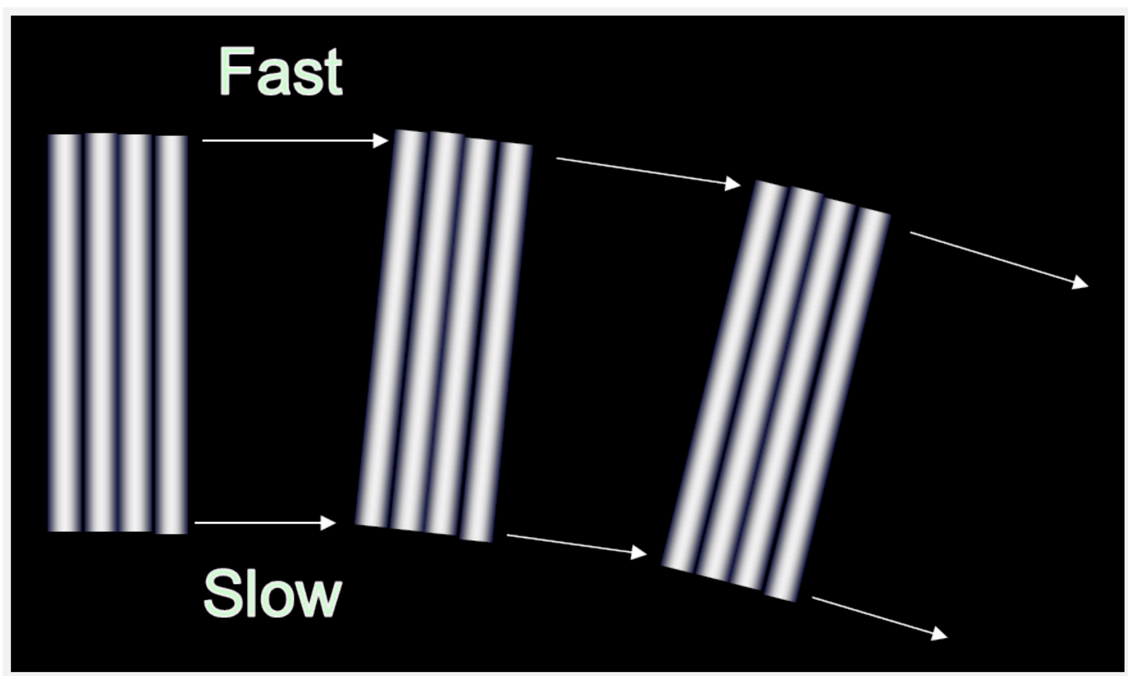

Figure 4.7 Waves refract toward the direction of slower wave speed. The rays are perpendicular to surfaces of constant phase.

4.2.1. Dispersion Relation and Metric Factors

Now consider the propagation of elastic waves in an static ideal elastic medium with non-uniform density. For soliton waves the dispersion relation can be written as:

$$\omega^2 = c^2 k^2 + M^2 \tag{4-4}$$

The dispersion relation relates the various sources of phase shifts in the wave (time derivatives and spatial derivatives). The mass term represents the contribution of convection and rotation to the frequency, whereas ck represents the contribution of restoring forces or torques in the medium, resulting in wave propagation at the reduced frequency $\omega' = ck = \left(\omega^2 - M^2\right)^{1/2}$. In terms of the reduced frequency the dispersion relation appears to represent ordinary wave propagation:

$$\omega'^2 = c^2 k^2 \tag{4-5}$$

The condition of constant phase is:

$$\omega' dt - \mathbf{k} \cdot d\mathbf{l} = 0 \tag{4-6}$$

This equation describes wave propagation at speed c (since $dl/dt = \omega'/k = c$) with $d\mathbf{l}$ parallel to $\mathbf{k}$, as distinct from convection and rotation. In other words, a disturbance evolves over time due to convection and rotation of the medium (resulting in mass) and wave propagation (momentum). The wave propagation occurs with the characteristic wave speed as described above, but convection and rotation increase the frequency, thereby raising the phase velocity and reducing the group velocity.

It is customary to assume positive frequency, in which case the relative sign of the wave vector may need to be altered:

$$\omega' dt = \pm \mathbf{k} \cdot d\mathbf{l} \tag{4-7}$$

The phase velocity ($v_\Phi \geq c$) is:

$$v_\Phi = \frac{dl}{dt} = \frac{\omega}{k} = \frac{\omega}{\left(\omega^2 - M^2\right)^{1/2}} c \tag{4-8}$$

The group velocity ($v_G \leq c$) is:

$$v_G = \frac{d\omega}{dk} = \frac{k}{\omega} c^2 = \frac{\left(\omega^2 - M^2\right)^{1/2}}{\omega} c \tag{4-9}$$

Using the propagation condition, we can define a 'phase separation' $d\chi$ for arbitrary space-time paths which measures the deviation from the propagation condition ($\omega' dt = \pm k dl$):

$$d\chi^2 = \omega'^2 dt^2 - k^2 dl^2$$

This equation could be modified by multiplying any non-zero variable on both sides. For example, we define the 'differential separation' ds by the relation:

Error! Objects cannot be created from editing field codes.

An alternative formulation yielding Fermat's principle can be found in Evans et al. [2001]. This differential separation should be zero for the true propagation path. The integrated separation is:

Error! Objects cannot be created from editing field codes. (4-10)

If the speed of light is variable, then neighboring space-time paths must still yield equal phase shifts in order to maintain the transverse orientation of the wave. This condition yields the equation of a geodesic:

$$\delta\int ds = \delta\int\left[\frac{c}{c_0}c_0^2 dt^2 - \frac{c_0}{c}dl^2\right]^{\frac{1}{2}} = 0 \tag{4-11}$$

This expression is equivalent to Einstein's formulation of general relativity [Einstein 1956 p. 78] if we assume a diagonal metric tensor with:

$$g^{tt} = \frac{c}{c_0}$$
$$g^{xx} = g^{yy} = g^{zz} = -\frac{c_0}{c} \tag{4-12}$$

So that the geodesic equation is:

$$\delta\int\left[\sum_{\mu\nu} g^{\mu\nu} dx_\mu dx_\nu\right]^{1/2} = 0 \tag{4-13}$$

With $dx_0 = c_0 dt$.

Hence Einstein's metric factors can be interpreted quite simply as the normalized values of the squared wave number and (reduced) frequency. Einstein's formulation is a bit more general in that it allows for non-isotropic metrics, and the above formula can be easily generalized to allow for independent variations of k_x/k_{0x}, k_y/k_{0y}, and k_z/k_{0z} (with appropriate dispersion relation). It is not clear that this generalization is important in nature, so it is not pursued here.

To simplify the integral, we introduce a parameter τ and rewrite the geodesic equation as:

$$\delta\int\left[\sum_{\mu\nu} g^{\mu\nu} \frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau}\right]^{1/2} d\tau \equiv \delta\int f^{1/2} d\tau = 0 \tag{4-14}$$

The Euler-Lagrange equations are:

$$f^{-1/2}\frac{d}{d\tau}\left[\sum_\nu g^{\alpha\nu}\frac{dx_\nu}{d\tau} + \sum_\nu g^{\mu\alpha}\frac{dx_\mu}{d\tau}\right] - f^{-1/2}\sum_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x_\alpha}\frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau} = 0 \tag{4-15}$$

Since the metric tensor is symmetric, this yields:

$$2\frac{d}{d\tau}\left[\sum_\nu g^{\alpha\nu}\frac{dx_\nu}{d\tau}\right] - \sum_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x_\alpha}\frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau} = 0 \tag{4-16}$$

We want to isolate $d^2x_\nu/d\tau^2$ in the above equation. This takes considerable effort to obtain:

$$\frac{d^2x_\nu}{d\tau^2} - \frac{1}{2}g^{\nu\alpha}\left[\frac{\partial g^{\alpha\beta}}{\partial x_\gamma} + \frac{\partial g^{\alpha\gamma}}{\partial x_\beta} - \frac{\partial g^{\beta\gamma}}{\partial x_\alpha}\right]\frac{dx_\beta}{d\tau}\frac{dx_\gamma}{d\tau} = 0 \tag{4-17}$$

For velocities small compared with the speed of light $\tau \approx c_0 t$ and $dt/d\tau \approx 1$, so to lowest order:

$$\frac{d^2x_i}{dt^2} = -\frac{c_0^2}{2}\frac{\partial g^{tt}}{\partial x_i} \tag{4-18}$$

The right hand side may be interpreted as gravitational acceleration and is equivalent to Einstein's expression [Einstein 1956 p.89] except for a different sign convention (Einstein uses imaginary time, thereby changing the sign of the temporal metric component).

4.2.2. Relation between metric components

For small changes in the speed of light with $c \approx c_0$:

$$\Delta\frac{c}{c_0} \approx -\Delta\frac{c_0}{c} \tag{4-19}$$

In terms of metric components:

$$\Delta g_{tt} \approx \Delta g_{xx} = \Delta g_{yy} = \Delta g_{zz} \approx \frac{\Delta c}{c_0} \tag{4-20}$$

This equality of first-order changes in metric components is in agreement with Einstein's result [Einstein 1956 p.89] except for the temporal sign convention. It is related to the fact that the Einstein tensor has zero divergence.

Using this result in the expression for gravitational acceleration yields:

$$\frac{d^2x_i}{dt^2} = -\frac{c_0}{2}\frac{\partial c}{\partial x_i} \approx -\frac{1}{4}\frac{\partial c^2}{\partial x_i} \tag{4-21}$$

Hence, to a first approximation, the gravitational acceleration is directly proportional to the gradient of the squared speed of light. The gravitational potential is:

$$U \approx \frac{c_0}{2}\Delta c \approx \frac{1}{4}\Delta c^2 = \frac{c^2 - c_0^2}{4} \tag{4-22}$$

where Δc^2 is the difference in the square of the speed of light from its unperturbed value. This expression for the gravitational potential is consistent with General Relativity [Einstein 1956 p. 84-93]. One may always offset this potential by a constant to make the values positive.

4.3. The gravitational potential

"The most incomprehensible thing about the universe is that it is comprehensible."
– Albert Einstein

The change in speed of a wave in an elastic solid may be attributed to either change of density or change of elastic constants. In this section we suppose that the change is due to compression. The

$$\frac{\partial^2}{\partial t^2}\rho = c_c^2\nabla^2\rho$$

equation of compression waves with speed c_c in an elastic solid is:

(4-23)

Assuming the density to be slowly varying allows the time derivatives to be neglected:

$$\nabla^2\rho = 0 \tag{4-24}$$

Many large massive objects are nearly spherical in shape, implying only a radial dependence:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\rho}{\partial r} = 0 \tag{4-25}$$

which has the solution:

$$\rho(r) = \rho_0\left[1+\frac{\eta}{r}\right] \tag{4-26}$$

where ρ_0 and η are constants. The speed of transverse waves is given by:

$$c^2(r) = \frac{\mu}{\rho} = \frac{\mu}{\rho_0\left[1+\frac{\eta}{r}\right]} \tag{4-27}$$

where μ is the shear modulus. The fractional variation of c^2 is given by:

$$\frac{\delta c^2}{c_0^2} = \frac{c^2(r)-c_0^2}{c_0^2} = \frac{1}{[1+\eta/r]} - 1 \approx -\frac{\eta}{r} + O\left(\frac{\eta}{r}\right)^2 \tag{4-28}$$

Hence the change of wave speed differs from the (1/*r*) dependence of the classical gravitational potential by the addition of higher order terms. However, even near the edge of the sun the variation is only $\delta c^2/c_0^2 \approx -10^{-6}$, so the second order difference is extremely small.

The change in the speed of light is evidently caused by the presence of mass (*M*) and falls off inversely proportional to distance (*r*) away from a spherically symmetrical distribution of mass (except for very small distances). The expression for the Newtonian gravitational potential is:

$$U(r) = \frac{GM}{r} \tag{4-29}$$

Where $G = 6.673 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$ is the gravitational constant. Notice that the gravitational potential has units of velocity squared.

4.4. Consequences of gravity

4.4.1. Newtonian gravity

Given the form of the gravitational potential and the expression for acceleration in terms of variations in the speed of light, we can express the gravitational acceleration of an object in terms of the potential:

$$\frac{d^2 x_i}{dt^2} = -\frac{1}{4} \frac{\partial c^2}{\partial x_i} = -\frac{\partial U}{\partial x_i} \tag{4-30}$$

The acceleration is simply equal to the gradient of the gravitational potential, as in Newtonian gravity.

4.4.2. Bending of light

For propagation of light waves, we can no longer neglect changes in position relative to changes in time. Take the velocity in the x_3 direction to be c and the gradient in the speed of light to be along x_1 as in
Figure 4.8.

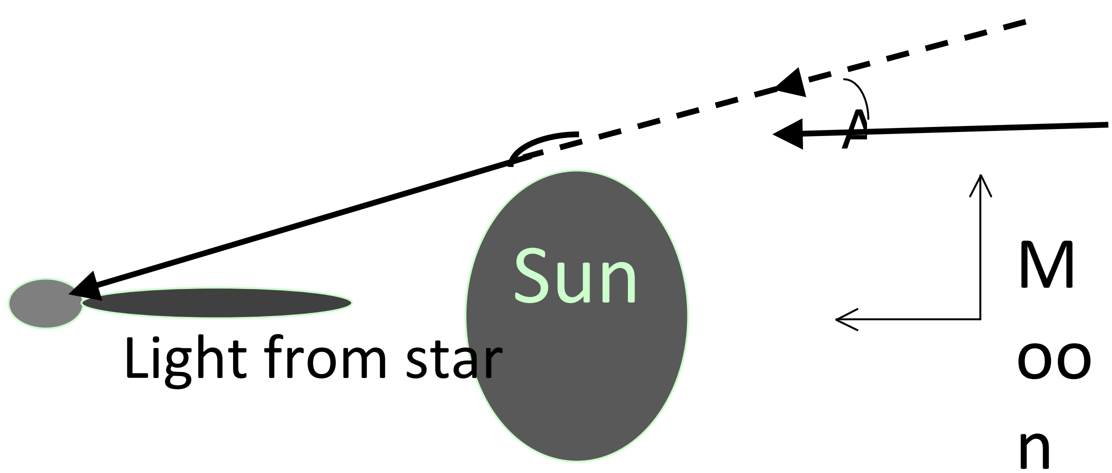

Figure 4.8 Apparent position of a star and the path of a light ray past the sun.

The acceleration is then:

$$\frac{d^2x_1}{dt^2} = \frac{1}{2}c^2 g_{11}\frac{\partial g_{00}}{\partial x_1} + \frac{1}{2}c^2 g_{11}\frac{\partial g_{33}}{\partial x_1} = -c\frac{\partial c}{\partial x_1} = -2\frac{\partial U}{\partial x_1} \tag{4-31}$$

This is twice the Newtonian acceleration rate.

Integrating over a path with a $1/r$ gravitational potential yields:

$$\begin{aligned} \frac{dx_1}{dt} &= -2\int_{-\infty}^{\infty} dt\left[\frac{\partial U}{\partial x_1}\right] = \frac{2}{c}\int_{-\infty}^{\infty} dx_3\left[\mu\frac{\partial}{\partial x_1}\left(\frac{1}{x_1^2 + x_3^2}\right)^{1/2}\right] \\ &= -\frac{2}{c}\int_{-\infty}^{\infty} dx_3\left[\mu\left(\frac{x_1}{\left(x_1^2 + x_3^2\right)^{3/2}}\right)\right] = -\frac{2}{c}\mu\left[\frac{x_3}{x_1\sqrt{x_1^2 + x_3^2}}\right]_{x_3=-\infty}^{x_3=\infty} = -\frac{4\mu}{cx_1} \end{aligned} \tag{4-32}$$

The gravitational coefficient for the sun is:

$$GM = \left(6.7\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)\left(2.0\times10^{30}\,\text{kg}\right) = 1.4\times10^{20}\,\frac{\text{m}^3}{\text{s}^2} \tag{4-33}$$

This yields a perpendicular velocity of:

$$\frac{dx_1}{dt} = -\frac{4\mu}{cx_1} = \frac{4\left(1.4\times10^{20}\,\text{m}^3/\text{s}^2\right)}{\left(3.0\times10^8\,\text{m}/\text{s}\right)x_1} = -\frac{1.9\times10^{12}\,\text{m}^2/\text{s}}{x_1} \tag{4-34}$$

Just outside the radius of the sun, $x_1 = 7.0\times10^8$ m:

$$\frac{dx_1}{dt} = -\frac{1.9\times10^{12}\,\text{m}^2/\text{s}}{7.0\times10^8\,\text{m}} = -2.7\times10^3\,\frac{\text{m}}{\text{s}} \tag{4-35}$$

The angle of deflection is the ratio between the deflection velocity and the speed of light:

$$\alpha = \frac{2.7\times10^3}{3.0\times10^8} = 9.0\times10^{-6}\,\text{radians} = 5.2\times10^{-4}\,\text{degrees} = 1.8'' \tag{4-36}$$

This deflection was first observed during a 1919 solar eclipse [Dyson, et al 1920]. More recent measurements use radio waves, which do not require waiting for eclipses [Lebach et al. 1995].

Since the light slows down near the sun, there is also a delay in the signal as compared with propagation in free space. This delay has also been measured and is in agreement with experiment [Shapiro et al. 1977, Bertotti et al. 2003].

4.4.3. Curvature of space

One supposedly bizarre prediction of general relativity is that "space is curved". What this means is that measurements of geometrical shapes are not consistent with Euclidean geometry. For example, suppose we measure the circumference of a circle of radius R_1 by shining light past a series of mirrors orbiting in space as shown in Figure 4.9. For simplicity, we will treat the earth as a point-like source of gravity.

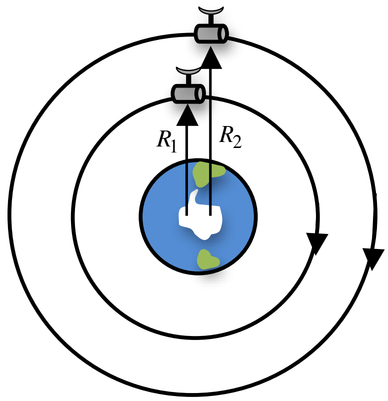

Figure 4.9 Distance measurements in a gravitational field.

We take the speed of light to be an approximation of the form derived above:

$$c(r) = c\frac{c_0}{\left[1+\eta/r\right]^{1/2}} \approx \frac{c_0}{1+\eta/2r} \tag{4-37}$$

Neglecting any delay during the reflection process, the light propagates with constant speed c_1 over a distance $2\pi R_1$, so that the propagation time is:

$$t_1 = \frac{2\pi R_1}{c_1} = \frac{2\pi}{c_0}\left[R_1 + \frac{\eta}{2}\right] \tag{4-38}$$

Since one cannot directly determine the absolute speed of light, the measured circumference L_1 is:

$$L_1 = c_0 t_1 = 2\pi\left[R_1 + \frac{\eta}{2}\right] \tag{4-39}$$

The circumference of a second circle with radius may be measured similarly. To avoid effects of different clock speeds, the transit time can be measured using the clock at R_1 by sending signals when the light wave is transmitted and when it is received by the satellite at R_2. The measured circumference of the circle at R_2 is:

$$L_2 = c_0 t_2 = 2\pi\left[R_2 + \frac{\eta}{2}\right] \tag{4-40}$$

The time of flight of light between the two circles is:

$$t_D = 2\int_{R_1}^{R_2} \frac{1}{c(r)}dr = \frac{2}{c_0}\int_{R_1}^{R_2}\left[1+\frac{\eta}{2r}\right]dr = \frac{2}{c_0}\left[R_2 - R_1 + \frac{\eta}{2}\ln\left[\frac{R_2}{R_1}\right]\right] \tag{4-41}$$

This means that the measured difference in radii is:

$$\Delta = \frac{c_0 t_D}{2} = R_2 - R_1 + \frac{\eta}{2}\ln\left[\frac{R_2}{R_1}\right] \tag{4-42}$$

According to Euclidean geometry, the two circumferences should be related by: $L_2 - L_1 = 2\pi\Delta$.

Instead, we have:

$$\frac{2\pi\Delta}{L_2 - L_1} = 1 + \frac{\eta}{2}\frac{\ln[R_2/R_1]}{(R_2 - R_1)} > 1 \tag{4-43}$$

Compared with Euclidean geometry, the measured circumference is smaller than expected for the measured diameter. This is the meaning of "curved space". However, the apparent curvature is actually attributable to the variation in the speed of light, which distorts the measurement of distances.

4.4.4. Black Holes

We saw above that light is deflected when it passes by a massive object such as the sun. If the gradient in the speed of light is large enough, then the light can become trapped. An object whose gravitational field is strong enough to trap light is called a "black hole".

For the geometry described above in
Figure 4.8 with variable x_1 replaced by *r* at the point of closest approach, the centripetal acceleration condition for trapping light is:

$$\frac{d^2 r}{dt^2} = -\frac{c^2(r)}{r} \tag{4-44}$$

In terms of the gravitational potential, this condition combined with (4-22) and (4-31) yields:

$$\frac{\partial U}{\partial r} = -\frac{4U(r) + c_0^2}{2r} \tag{4-45}$$

In terms of the mass of the black hole:

$$\frac{GM}{r^2} = \frac{4GM/r - c_0^2}{2r} = 2\frac{GM}{r^2} - \frac{c_0^2}{2r} \tag{4-46}$$

Solving for r:

$$r = \frac{2GM}{c_0^2} \tag{4-47}$$

This radius is called the “Schwarzchild radius”. Any light which reaches this point from the outside will be trapped.

Black holes were once considered an absurdity, but there is now a wealth of evidence for their existence in the universe.

4.4.5. Gravitomagnetism

Thus far we have regarded the gravitational potential, and the elastic aether, as being static but variable in space. This point of view cannot be valid for observers with arbitrary relative motion. If the gravitational disturbance propagates through space with a speed different than that of light, then it could be possible to determine the absolute rest frame of the aether from the directional changes in the apparent velocity of the gravitational disturbance (gravity waves). However, if the gravitational disturbance propagates with the same speed as light waves, then it will conform to the ordinary Lorentz transformations and there would be no way to determine an absolute reference frame. We will assume that this is the case.

Relative motion of the source of a gravitational disturbance (change of elastic aether) does affect wave propagation. Just as the magnetic force can be attributed to the Lorentz contraction of a moving line of charged particles, a magnetic ‘coriolis’ force may be attributed to Lorentz contraction of moving matter.

Start from a reference frame with static gravitational sources:

$$\begin{aligned} g^{t't'} &= 1 + \frac{\Delta c}{c_0} \\ g^{x'x'} &= g^{y'y'} = g^{z'z'} = -1 + \frac{\Delta c}{c_0} \\ g^{t'x'} &= g^{x't'} = 0 \end{aligned} \qquad (4\text{-}48)$$

Applying Lorentz transformations to the second index yields:

$$\begin{aligned} g^{t't} &= \gamma g^{t't'} + \beta\gamma g^{t'x'} = \gamma\left(1 + \frac{\Delta c}{c_0}\right) \\ g^{x'x} &= \gamma g^{x'x'} + \beta\gamma g^{x't'} = \gamma\left(-1 + \frac{\Delta c}{c_0}\right) \\ g^{t'x} &= \gamma g^{t'x'} + \beta\gamma g^{t't'} = \beta\gamma\left(1 + \frac{\Delta c}{c_0}\right) \\ g^{x't} &= \gamma g^{x't'} + \beta\gamma g^{x'x'} = \beta\gamma\left(-1 + \frac{\Delta c}{c_0}\right) \end{aligned} \qquad (4\text{-}49)$$

And applying Lorentz transformations to the first index then yields:

$$g^{tt} = \gamma g^{t't} + \beta\gamma g^{x't} = \gamma^2\left(1+\frac{\Delta c}{c_0}\right) + \beta^2\gamma^2\left(-1+\frac{\Delta c}{c_0}\right) = \left(1+\frac{\Delta c}{c_0}\right) + O\left(\beta^2\right)$$

$$g^{xx} = \gamma g^{x'x} + \beta\gamma g^{t'x} = \gamma^2\left(-1+\frac{\Delta c}{c_0}\right) + \beta^2\gamma^2\left(1+\frac{\Delta c}{c_0}\right) = \left(-1+\frac{\Delta c}{c_0}\right) + O\left(\beta^2\right)$$

$$g^{tx} = \gamma g^{t'x} + \beta\gamma g^{x'x} = \beta\gamma^2\left(1+\frac{\Delta c}{c_0}\right) + \beta\gamma^2\left(-1+\frac{\Delta c}{c_0}\right) = 2\beta\frac{\Delta c}{c_0} + O\left(\beta^2\right)$$

$$g^{xt} = \gamma g^{x't} + \beta\gamma g^{t't} = \beta\gamma^2\left(-1+\frac{\Delta c}{c_0}\right) + \beta\gamma^2\left(1+\frac{\Delta c}{c_0}\right) = 2\beta\frac{\Delta c}{c_0} + O\left(\beta^2\right) \tag{4-50}$$

Moving gravitational sources therefore introduce non-diagonal components of the metric tensor. For motion in the *x*-direction, the metric components to first order in $\beta = v/c$ are:

$$g_{tt} = 1+\frac{\Delta c}{c_0}$$

$$\Delta g_{xx} = \Delta g_{yy} = \Delta g_{zz} = -1+\frac{\Delta c}{c_0}$$

$$\Delta g_{ti} = \Delta g_{it} = 2\frac{v_i}{c_0}\frac{\Delta c}{c_0} \tag{4-51}$$

For example, the gravitational potential of an infinite rod moving with speed v_x parallel to its axis may appear to be static, but any perturbation would actually propagate along with the rod. The result is that the gravitational deflection of an object is increased if the relative velocity between it and the gravitational source is increased. This is called the gravitomagnetic, or frame-dragging, effect. The existence of this effect has apparently been confirmed in experiments, but doubts about accuracy remain [Iorio 2009].

4.5. Summary

"*The bigger they are the harder they fall.*"
– Anonymous

The above analysis demonstrates that gravity can be interpreted as wave refraction in a non-uniform medium. Unlike quantum theories in which gravity waves are assigned a spin of 2, the present model treats gravity as a scalar associated with changes in either density or elasticity of the solid aether. There is absolutely no physical evidence indicating that gravity should be quantized.

Compression waves in a solid can in principle propagate at a speed equal or greater than the speed of transverse (or torsion) waves. Therefore it is quite possible that gravity waves propagate at a speed greater than *c*. If that is the case then the measured speed would also be direction dependent due to the earth's motion relative to the vacuum. Such a difference in wave speeds would also be evident in gravitomagnetic effects.

In summary, gravity may be interpreted as a description of wave refraction due to decreased velocity of light in the vicinity of matter. If the aether is taken to be an elastic solid, then the variation in light speed might be attributed to compression or change in elasticity. The spatial metric components are interpreted as the ratio between the squared wave numbers at different positions. The temporal metric component is interpreted similarly as the ratio between squared frequencies at different positions. Conservation of angular momentum and energy yield the correct relation between spatial and temporal metric components. The derived form of the gravitational potential falls off as $1/r$ for large distances but also includes higher-order terms.

Gravitation deflects light in accordance with the laws of wave refraction. It also makes space appear to be non-Euclidean. Black holes bend light rays so strongly that the light becomes trapped. All of these effects are easily understood using the classical model of an elastic solid aether.

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Figures

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Figure 4.1 Isaac Newton (1643 – 1727).
Source: http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Newton.html

Figure 4.2 Pierre-Simon Laplace (1749-1827).
Source: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Laplace.html

Figure 4.3 Lóránd Baron von Eötvös (1848-1919).
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Figure 4.4 Albert Einstein (1879 – 1955).
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Figure 4.5 David Hilbert (1862 – 1943).
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Figure 4.6 Karl Schwarzschild (1873-1916).
Source: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Schwarzschild.html

Epilogue

We have seen that rotational, or torsion, waves in an elastic solid obey the same basic equation as matter waves: the Dirac equation. With appropriate dimensional scaling of the wave function, all dynamical quantities (momentum, angular momentum, and energy) have consistent mathematical expressions for both torsion waves and matter waves. Although we have not yet found specific soliton solutions for elementary particles, we are clearly on the right track. Even if the specific model proposed here proves to be inadequate, it should be possible to modify the theory to achieve agreement. Therefore the model of the vacuum as an elastic solid, even if incomplete or inaccurate, should still be a valuable tool for determining the actual structure of the universe.

Torsion solitons seem to fit the requirements of wave-particle duality. If the internal structure is unchanged by an interaction (elastic collision) then the wave packet can be treated as a point 'particle'. However, they are pure waves and hence subject to uncertainty principles. The theory predicts the existence of both matter and anti-matter since any solutions must come in pairs which are related by conjugation of handedness at each point.

Recall that nature appears to be symmetric with respect to conjugation of both left and right handedness and matter and anti-matter, but not either one separately. The torsion wave interpretation of the Dirac equation explicitly attributes handedness to matter and anti-matter, thereby restoring the intuitive expectation of mirror symmetry of physical phenomena.

We have seen that soliton wave correlations are computed in the same manner as for quantum mechanics.

The combination of soliton waves is very familiar in nature: atoms are formed by combining protons, neutrons, and electrons. Quantum mechanics (and quantum field theory) has yielded remarkable success in predicting the likelihood of various outcomes from particle interactions. However, quantum theory cannot be regarded as complete unless it can describe what happens during an interaction.

A popular cartoon by Sidney Harris [www.sciencecartoonsplus.com/pages/gallery.php] shows a professor deriving a physical result on an equation-filled blackboard. In the middle of the derivation is written, "then a miracle occurs." Although intended as a joke, this cartoon aptly describes the present state of quantum theory. A photon approaches an atom, then disappears as the atom jumps from one energy level to another. No one can say what was happening while the photon was disappearing and the atomic energy level was jumping. The theory is incomplete and must rely on a miracle.

Rotational, or torsion, waves in an elastic solid should have solutions which satisfy a Klein-Gordon equation in each dimension, with as many as three different masses attributed to a single soliton solution. The existence of three coupled equations suggests an explanation for various types of elementary particles: leptons with one mass term, mesons with two mass terms of equal value, and baryons with three mass terms (quarks). Perhaps these are superficial analogies, but we won't know until the classical solutions are studied.

We saw that interactions of torsion solitons involve real-valued potentials. This is closely analogous to the Standard Model, which "asserts that the material in the universe is made up of elementary fermions interacting through fields, of which they are the sources. The particles associated with the interaction fields are bosons" [Cottingham and Greenwood 1998]. In standard quantum mechanics the potential is assumed to be directly associated with particles. In the torsion interpretation, the 'particles' associated with the potentials represent wave packets whose absorption or emission change the soliton state of the 'source particle'.

There is a certain logical problem with the idea that fermions are the 'source' of boson fields, for in fact the fermions cannot be separated from the surrounding fields. In particular, since an electron always has long-range electromagnetic potentials associated with it, there is no logical justification for asserting it to be a 'point particle'. On the contrary, it seems more sensible to ascribe all of the properties of the electron to the Dirac wave function from which the potentials are derived.

We now understand that the laws of special relativity are simply a consequence of the wave nature of matter and are not evidence of any intrinsic geometric relationship between space and time. The apparent constancy of the speed of light with respect to moving observers is due to the simple undisputed fact that matter waves travel the same speed as light waves (eigenvalues of the Dirac velocity operator have magnitude c). The negative result of the Michelson-Morley experiment did not disprove the notion that the vacuum is a medium for wave propagation. It simply confirmed that light and matter waves have the same wave speed.

The equations of general relativity are also exactly what we would expect to find if matter waves refract in an inhomogeneous medium. Instead of saying that space is curved, we can with equal validity say that rulers change length in a gravitational field. Since we expect torsion to be accompanied by a slight compression, the torsion interpretation of matter offers a simple explanation of why the presence of matter would lower the wave speed in its vicinity. One key prediction of the torsion hypothesis is that the frequency and wave number vary inversely (the fractional changes are equal and opposite for small variations), in agreement with general relativity.

One principle in formulating scientific theories is the application of "Ockham's razor". Any unnecessary complexity should be cut out of the theory so that it is as simple as possible. If there are two explanations for some phenomenon, the simplest one is generally preferred. Modern physicists have made great strides toward describing nature. Einstein's theories of relativity correctly predict the refraction of light near massive objects and the influence of gravity on planets. The 'Standard Model' seems to offer a mathematical description of particle interactions which is consistent with all experimental results. New theories of strings and supersymmetries hold great promise for incorporating gravity into the quantum mechanical framework. But there is a problem with modern physics. The problem is that while work progressed on complicated problems of non-Euclidean geometry and higher-dimensional spaces, simpler problems were left unsolved: What is the classical expression for wave angular momentum? What relativity principle would result if matter consisted of waves? What is the mathematical expression for torsion? How are measurements of matter limited by the fact that the instruments of measurement also consist of matter? These basic questions should be answered before delving into the complex mathematics of modern physical theories. Given that the physical properties of the vacuum are unknown, one can hardly expect to understand matter waves without first being able to understand waves in a simple medium with known properties: a uniform, isotropic elastic solid.

Imagine what our interpretation of modern physics would be if Paul Dirac had understood torsion waves in 1928. Discovering matter to obey the same equations, with the same dynamical operators, he would have had every reason to think that matter consists of torsion waves. Any suggestion that matter waves were actually 'probability waves' with no other physical interpretation would have been preposterous. Even without such knowledge, Dirac was convinced that the Hamiltonian nature of the equations requires the existence of an aether.

It is often said that extraordinary claims require extraordinary evidence. No one could seriously claim to have an interpretation of quantum mechanics without producing an accurate derivation of some basic physical quantities such as the fine structure constant or relative masses of particles. At this time no such derivation has been made, and the proposal that matter consists of torsion waves must be regarded as unproven. Nonetheless, we have shown that this simple model yields mathematical equations which cannot be distinguished from the equations for material particles without further detailed analysis. A wide range of 'non-classical' phenomena are in fact completely consistent with classical notions of physics. With this logical framework in hand we can now hope to truly understand the universe, and not merely describe it.

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Appendix A: Angular Eigenfunctions

Isolate the angular derivatives:

$$[\nabla\cdot\boldsymbol{\sigma}]\psi = \sigma_r\left[\frac{\partial}{\partial r} + i\frac{\boldsymbol{\sigma}}{r}\cdot[\mathbf{r}\times\nabla]\right]\psi = \sigma_r\left[\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\left[\sigma_\phi\frac{\partial}{\partial\theta} - \sigma_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\right]\psi \tag{A-1}$$

Separation of the equation requires:

$$\left[\sigma_\phi\frac{\partial}{\partial\theta} - \sigma_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\psi = K\psi \tag{A-2}$$

where K is a constant matrix.

Let:

$$\psi = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{A-3}$$

so that each two-component wave function satisfies:

$$\left[\sigma_\phi\frac{\partial}{\partial\theta} - \sigma_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\psi_\alpha = \mathrm{i}\,l_\alpha\psi_\alpha \tag{A-4}$$

The two-component $\boldsymbol{\sigma}$ matrices in polar coordinates are:

$$\sigma_r = \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta\end{pmatrix}$$

$$\sigma_\theta = \begin{pmatrix}-\sin\theta & \cos\theta\, e^{-i\phi}\\ \cos\theta\, e^{i\phi} & \sin\theta\end{pmatrix}$$

$$\sigma_\phi = \begin{pmatrix}0 & -\mathrm{i}\, e^{-i\phi}\\ \mathrm{i}\, e^{i\phi} & 0\end{pmatrix} \tag{A-5}$$

Note that:

$$\frac{\partial}{\partial\phi}\sigma_\phi = \begin{pmatrix} 0 & -e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} = -\sin\theta\sigma_r - \cos\theta\sigma_\theta \tag{A-6}$$

Expand (A-**4**):

$$\left[\begin{pmatrix} 0 & -\mathrm{i}\, e^{-\mathrm{i}\phi} \\ \mathrm{i}\, e^{\mathrm{i}\phi} & 0 \end{pmatrix}\frac{\partial}{\partial\theta} - \begin{pmatrix} -\sin\theta & \cos\theta\, e^{-\mathrm{i}\phi} \\ \cos\theta\, e^{\mathrm{i}\phi} & \sin\theta \end{pmatrix}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\psi = \mathrm{i}\, l_\alpha\psi \tag{A-7}$$

Examine the ϕ dependence for $\psi = [f, g]^T$:

$$\begin{aligned} &-\mathrm{i}\, e^{-i\phi}\frac{\partial g}{\partial\theta} + \frac{\partial f}{\partial\phi} - \frac{\cos\theta}{\sin\theta}e^{-i\phi}\frac{\partial g}{\partial\phi} = \mathrm{i}\, l_\alpha f \\ &\mathrm{i}\, e^{i\phi}\frac{\partial f}{\partial\theta} - \frac{\cos\theta}{\sin\theta}e^{i\phi}\frac{\partial f}{\partial\phi} - \frac{\partial g}{\partial\phi} = \mathrm{i}\, l_\alpha g \end{aligned} \tag{A-8}$$

For an exponential dependence $f \sim \exp(\mathrm{i}\, m\phi)$, the two-component wave function must have the form:

$$\psi_\alpha = f_\alpha(r)\begin{bmatrix} \Phi_1(\theta)\exp(\mathrm{i}\, m\phi) \\ \Phi_2(\theta)\exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \tag{A-9}$$

Let:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \Phi_1(\theta)\exp(\mathrm{i}\, m\phi) \\ \Phi_2(\theta)\exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \tag{A-10}$$

Then:

$$\begin{aligned} &-\mathrm{i}\frac{\partial}{\partial\theta}\Phi_2 + \mathrm{i}\, m\Phi_1 - \frac{\mathrm{i}[m+1]\cos\theta}{\sin\theta}\Phi_2 = \mathrm{i}\, l_\alpha\Phi_1 \\ &\mathrm{i}\frac{\partial}{\partial\theta}\Phi_1 - \frac{\mathrm{i}\, m\cos\theta}{\sin\theta}\Phi_1 - \mathrm{i}[m+1]\Phi_2 = \mathrm{i}\, l_\alpha\Phi_2 \end{aligned} \tag{A-11}$$

Regroup:

$$\begin{aligned} &\left[-\frac{\partial}{\partial\theta} - \frac{[m+1]\cos\theta}{\sin\theta}\right]\Phi_2 + [m - l_\alpha]\Phi_1 = 0 \\ &\left[\frac{\partial}{\partial\theta} - \frac{m\cos\theta}{\sin\theta}\right]\Phi_1 - [[m+1] + l_\alpha]\Phi_2 = 0 \end{aligned} \tag{A-12}$$

Substitute:

$$\left[-\frac{\partial}{\partial\theta}-\frac{[m+1]\cos\theta}{\sin\theta}\right]\left[\frac{\partial}{\partial\theta}-\frac{m\cos\theta}{\sin\theta}\right]\Phi_1=[l_\alpha-m][l_\alpha+[m+1]]\Phi_1$$
$$\left[\frac{\partial}{\partial\theta}-\frac{m\cos\theta}{\sin\theta}\right]\left[-\frac{\partial}{\partial\theta}-\frac{[m+1]\cos\theta}{\sin\theta}\right]\Phi_2=[l_\alpha+[m+1]][l_\alpha-m]\Phi_2 \quad \text{(A-13)}$$

The equation for Φ_1 is:

$$\frac{\partial^2}{\partial\theta^2}\Phi_1+\left[\frac{\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\theta}\Phi_1-\left[\frac{m^2\cos^2\theta}{\sin^2\theta}-m\right]\Phi_1=-[l_\alpha-m][l_\alpha+[m+1]]\Phi_1$$

$$\frac{\partial^2}{\partial\theta^2}\Phi_1+\left[\frac{\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\theta}\Phi_1-\left[\frac{m^2\cos^2\theta}{\sin^2\theta}-m\right]\Phi_1=-\{l_\alpha[l_\alpha+1]-m[m+1]\}\Phi_1 \quad \text{(A-14)}$$

$$\frac{\partial^2}{\partial\theta^2}\Phi_1+\left[\frac{\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\theta}\Phi_1-\frac{m^2}{\sin^2\theta}\Phi_1=-l_\alpha[l_\alpha+1]\Phi_1 \quad \text{(A-15)}$$

Let:

$$x=\cos\theta$$
$$\frac{\partial}{\partial\theta}\Phi_1=-\sin\theta\frac{\partial}{\partial x}\Phi_1=-\left[1-x^2\right]^{1/2}\frac{\partial}{\partial x}\Phi_1$$
$$\frac{\partial^2}{\partial\theta^2}\Phi_1=\sin\theta\frac{\partial}{\partial x}\left[\left[1-x^2\right]^{1/2}\frac{\partial}{\partial x}\Phi_1\right]=-x\frac{\partial}{\partial x}\Phi_1+\left[1-x^2\right]\frac{\partial^2}{\partial x^2}\Phi_1 \quad \text{(A-16)}$$

In terms of x:

$$\left[1-x^2\right]\frac{\partial^2}{\partial x^2}\Phi_1-2x\frac{\partial}{\partial x}\Phi_1+\left[l_\alpha[l_\alpha+1]-\frac{m^2}{1-x^2}\right]\Phi_1=0 \quad \text{(A-17)}$$

which is the Legendre's associated differential equation. Solutions with arbitrary amplitude *a* bounded in the interval $-1\le x\le 1$ may be denoted:

$$\Phi_1=aP_{l_\alpha}^m(x)=a[-1]^m\frac{\left[1-x^2\right]^{m/2}}{2^{l_\alpha}l_\alpha!}\frac{d^{m+l_\alpha}}{dx^{m+l_\alpha}}\left(x^2-1\right)^{l_\alpha} \quad \text{(A-18)}$$

although the reader should be wary of differing sign conventions. The normalized spherical harmonics are:

$$Y_{l_\alpha}^{m} = \sqrt{\frac{2l_\alpha + 1}{4\pi}\frac{[l_\alpha - m]!}{[l_\alpha + m]!}}P_{l_\alpha}^{m}(\cos\theta)\exp(\mathrm{i}\, m\phi) \tag{A-19}$$

The equation for Φ_2 is obtained by replacing m with $-(m+1)$:

$$\frac{\partial^2}{\partial\theta^2}\Phi_2 + \left[\frac{\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\theta}\Phi_2 - \frac{[m+1]^2}{\sin^2\theta}\Phi_2 = -l_\alpha[l_\alpha + 1]\Phi_2 \tag{A-20}$$

Or in terms of x:

$$[1 - x^2]\frac{\partial^2}{\partial x^2}\Phi_2 - 2x\frac{\partial}{\partial x}\Phi_2 + \left[l_\alpha[l_\alpha + 1] - \frac{[m+1]^2}{1 - x^2}\right]\Phi_2 = 0 \tag{A-21}$$

The solution with arbitrary amplitude b is:

$$\Phi_2 = bP_{l_\alpha}^{m}(x) = b[-1]^m \frac{[1 - x^2]^{m/2}}{2^{l_\alpha} l_\alpha!}\frac{d^{m+l_\alpha}}{dx^{m+l_\alpha}}(x^2 - 1)^{l_\alpha} \tag{A-22}$$

The second normalized spherical harmonic is:

$$Y_{l_\alpha}^{m+1} = \sqrt{\frac{2l_\alpha + 1}{4\pi}\frac{[l_\alpha - m - 1]!}{[l_\alpha + m + 1]!}}P_{l_\alpha}^{m+1}(\cos\theta)\exp(\mathrm{i}[m+1]\phi) \tag{A-23}$$

The wave function may now be written as:

$$\psi_\alpha = \sum_{l_\alpha, m} f_{lm}(r)\begin{bmatrix} aY_{l_\alpha}^{m} \\ bY_{l_\alpha}^{m+1}\end{bmatrix} \tag{A-24}$$

The ratio of the two solutions can be determined from the original equations:

$$\left[\frac{\partial}{\partial\theta} - \frac{m\cos\theta}{\sin\theta}\right]\sqrt{\frac{2l_\alpha + 1}{4\pi}\frac{[l_\alpha - m]!}{[l_\alpha + m]!}}aP_{l_\alpha}^{m} - [[m+1] + l_\alpha]b\sqrt{\frac{2l_\alpha + 1}{4\pi}\frac{[l_\alpha - m - 1]!}{[l_\alpha + m + 1]!}}P_{l_\alpha}^{m+1} = 0 \tag{A-25}$$

Comparison with the identity:

$$P_l^{m+1} = \frac{d}{d\theta}P_l^m - m\frac{\cos\theta}{\sin\theta}P_l^m \tag{A-26}$$

yields:

$$\frac{b}{a} = \frac{\sqrt{[l_\alpha - m]}}{\sqrt{l_\alpha + m + 1}} \tag{A-27}$$

The normalized angular functions are:

$$\psi_\alpha = \begin{bmatrix} \sqrt{\frac{l_\alpha + m + 1}{2l_\alpha + 1}} Y_{l_\alpha}^{m} \\ \sqrt{\frac{[l_\alpha - m]}{2l_\alpha + 1}} Y_{l_\alpha}^{m+1} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{[l_\alpha + m + 1]}{4\pi} \frac{[l_\alpha - m]!}{[l_\alpha + m]!}} P_{l_\alpha}^{m} \exp(\mathrm{i}\, m\phi) \\ \sqrt{\frac{1}{4\pi} \frac{[l_\alpha - m]!}{[l_\alpha + m + 1]!}} P_{l_\alpha}^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix}$$

$$= N_\alpha \begin{bmatrix} [l_\alpha + m + 1] P_{l_\alpha}^{m} \exp(\mathrm{i}\, m\phi) \\ P_{l_\alpha}^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \tag{A-28}$$

These solutions are multiplied by the matrix σ_r in the bispinor equation.

$$\sigma_r \psi_\alpha = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} N_\alpha \begin{bmatrix} [l_\alpha + m + 1] P_{l_\alpha}^{m} \exp(\mathrm{i}\, m\phi) \\ P_{l_\alpha}^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \left[[l_\alpha + m + 1] P_{l_\alpha}^{m} \cos\theta + P_{l_\alpha}^{m+1} \sin\theta\right] \exp(\mathrm{i}\, m\phi) \\ \left[-P_{l_\alpha}^{m+1} \cos\theta + [l_\alpha + m + 1] P_{l_\alpha}^{m} \sin\theta\right] \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \tag{A-29}$$

Compare this with the identities:

$$P_{l+1}^{m} = -[l + m] \sin\theta P_l^{m-1} + \cos\theta P_l^{m} \tag{A-30}$$

and:

$$[l + m][l - m + 1] \sin\theta P_l^{m-1} = -2m \cos\theta P_l^{m} - \sin\theta P_l^{m+1} \tag{A-31}$$

which combine to yield:

$$[l - m + 1] P_{l+1}^{m} = [l + m + 1] \cos\theta P_l^{m} + \sin\theta P_l^{m+1} \tag{A-32}$$

This identity applies to the top element. For the lower element use the identity:

$$P_{l+1}^{m+1} = -[l + m + 1] \sin\theta P_l^{m} + \cos\theta P_l^{m+1} \tag{A-33}$$

The result is:

$$\sigma_r \psi_\alpha = N_\alpha \begin{bmatrix} [l+1-m] P_{l+1}^m \exp(\mathrm{i}\, m\phi) \\ - P_{l+1}^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} = \begin{bmatrix} \sqrt{\dfrac{[l+1-m]}{4\pi} \dfrac{[l+1-m]!}{[l+1+m]!}} P_{l+1}^m \exp(\mathrm{i}\, m\phi) \\ - \sqrt{\dfrac{1}{4\pi} \dfrac{[l-m]!}{[l+1+m]!}} P_{l+1}^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \tag{A-34}$$

In terms of Y's:

$$\sigma_r \psi_\alpha = \begin{bmatrix} \sqrt{\dfrac{[l+1-m]}{2[l+1]+1}} Y_{l+1}^m \\ - \sqrt{\dfrac{[l+1]+1+m}{2[l+1]+1}} Y_{l+1}^{m+1} \end{bmatrix} \tag{A-35}$$

Letting $l_2 = l+1$ yields:

$$\sigma_r \psi_\alpha \equiv \psi_\beta = \begin{bmatrix} \sqrt{\dfrac{[l_2 - m]}{2l_2 + 1}} Y_{l_2}^m \\ - \sqrt{\dfrac{l_2 + 1 + m}{2l_2 + 1}} Y_{l_2}^{m+1} \end{bmatrix} = N_R \begin{bmatrix} [l+1-m] P_{l+1}^m \exp(\mathrm{i}\, m\phi) \\ - P_{l+1}^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \tag{A-36}$$

These solutions have the property:

$$\begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{L} \psi_\alpha &= \boldsymbol{\sigma} \cdot [\mathbf{r} \times \mathrm{i} \nabla] \psi_\alpha = \mathrm{i} \left[\sigma_\phi \frac{\partial}{\partial \theta} - \sigma_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right] \psi_\alpha \\ &= -\mathrm{i} \begin{pmatrix} 0 & -\mathrm{i} e^{-\mathrm{i}\phi} \\ \mathrm{i} e^{\mathrm{i}\phi} & 0 \end{pmatrix} N_\alpha \sin\theta \begin{bmatrix} [l+m+1] P_l'^m \exp(\mathrm{i}\, m\phi) \\ P_l'^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \\ &\quad - \mathrm{i} \begin{pmatrix} -\sin\theta & \cos\theta e^{-\mathrm{i}\phi} \\ \cos\theta e^{\mathrm{i}\phi} & \sin\theta \end{pmatrix} \frac{N_\alpha}{\sin\theta} \begin{bmatrix} \mathrm{i}\, m[l+m+1] P_l^m \exp(\mathrm{i}\, m\phi) \\ \mathrm{i}[m+1] P_l^{m+1} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \\ &= \frac{N_\alpha}{\sin\theta} \begin{bmatrix} \left\{ -\sin^2\theta P_l'^{m+1} - m[l+m+1] \sin\theta P_l^m + [m+1] \cos\theta P_l^{m+1} \right\} \exp(\mathrm{i}\, m\phi) \\ \left\{ [l+m+1] \sin^2\theta P_l'^m + m[l+m+1] \cos\theta P_l^m + [m+1] \sin\theta P_l^{m+1} \right\} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \\ &= \frac{N_\alpha}{\sin\theta} \begin{bmatrix} \left\{ -[l+m+1][l-m] \sin\theta P_l^m - m[l+m+1] \sin\theta P_l^m \right\} \exp(\mathrm{i}\, m\phi) \\ \left\{ -[l+m+1] \sin\theta P_l^{m+1} + [m+1] \sin\theta P_l^{m+1} \right\} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} \\ &= -l N_\alpha \begin{bmatrix} \left\{ [l+m+1] P_l^m \right\} \exp(\mathrm{i}\, m\phi) \\ \left\{ P_l^{m+1} \right\} \exp(\mathrm{i}[m+1]\phi) \end{bmatrix} = -l N_\alpha \psi_\alpha \end{aligned} \tag{A-37}$$

$$
\begin{aligned}
\boldsymbol{\sigma}\cdot\mathbf{L}\psi_\beta &= \boldsymbol{\sigma}\cdot[\mathbf{r}\times\mathrm{i}\nabla]\psi_\beta = \mathrm{i}\left[\sigma_\phi\frac{\partial}{\partial\theta}-\sigma_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\psi_\beta \\
&= -\mathrm{i}\begin{pmatrix}0 & -\mathrm{i}e^{-\mathrm{i}\phi}\\ \mathrm{i}e^{\mathrm{i}\phi} & 0\end{pmatrix} N_\alpha\sin\theta\begin{bmatrix}[l-m+1]P_{l+1}'^{m}\exp(\mathrm{i}m\phi)\\ -P_{l+1}'^{m+1}\exp(\mathrm{i}[m+1]\phi)\end{bmatrix} \\
&\quad -\mathrm{i}\begin{pmatrix}-\sin\theta & \cos\theta e^{-\mathrm{i}\phi}\\ \cos\theta e^{\mathrm{i}\phi} & \sin\theta\end{pmatrix}\frac{N_\alpha}{\sin\theta}\begin{bmatrix}\mathrm{i}m[l-m+1]P_{l+1}^{m}\exp(\mathrm{i}m\phi)\\ -\mathrm{i}[m+1]P_{l+1}^{m+1}\exp(\mathrm{i}[m+1]\phi)\end{bmatrix} \\
&= \frac{N_\alpha}{\sin\theta}\begin{bmatrix}\left\{\sin^2\theta P_{l+1}'^{m+1}-m[l-m+1]\sin\theta P_{l+1}^{m}-[m+1]\cos\theta P_{l+1}^{m+1}\right\}\exp(\mathrm{i}m\phi)\\ \left\{[l-m+1]\sin^2\theta P_{l+1}'^{m}+m[l-m+1]\cos\theta P_{l+1}^{m}-[m+1]\sin\theta P_{l+1}^{m+1}\right\}\exp(\mathrm{i}[m+1]\phi)\end{bmatrix} \\
&= \frac{N_\alpha}{\sin\theta}\begin{bmatrix}\left\{[l+m+2][l-m+1]\sin\theta P_{l+1}^{m}-m[l-m+1]\sin\theta P_{l+1}^{m}\right\}\exp(\mathrm{i}m\phi)\\ \left\{-[l-m+1]\sin\theta P_{l+1}^{m+1}-[m+1]\sin\theta P_{l+1}^{m+1}\right\}\exp(\mathrm{i}[m+1]\phi)\end{bmatrix} \\
&= N_\alpha\begin{bmatrix}\left\{[l+2][l-m+1]P_{l+1}^{m}\right\}\exp(\mathrm{i}m\phi)\\ \left\{-[l+2]P_{l+1}^{m+1}\right\}\exp(\mathrm{i}[m+1]\phi)\end{bmatrix} = [l+2]N_\alpha\psi_\beta
\end{aligned}
\tag{A-38}
$$

Letting $\kappa = l+1$, we can write the convection term as:

$$
\begin{aligned}
[\boldsymbol{\sigma}\cdot\nabla]F\psi_\alpha &= \sigma_r\left[\frac{\partial}{\partial r}-\frac{l}{r}\right]F\psi_\alpha = \sigma_r\left[\frac{\partial}{\partial r}+\frac{1}{r}-\frac{\kappa}{r}\right]F\psi_\alpha \\
[\boldsymbol{\sigma}\cdot\nabla]G\psi_\beta &= \sigma_r\left[\frac{\partial}{\partial r}+\frac{l+2}{r}\right]G\psi_\beta = \sigma_r\left[\frac{\partial}{\partial r}+\frac{1}{r}+\frac{\kappa}{r}\right]G\psi_\beta
\end{aligned}
\tag{A-39}
$$

Or, more concisely:

$$
[\boldsymbol{\sigma}\cdot\nabla]\psi = \sigma_r\left[\frac{\partial}{\partial r}+\frac{1}{r}-\beta_3\frac{\kappa}{r}\right]\psi
\tag{A-40}
$$

Appendix B: Lorentz Transformations

Let:

$$
\begin{aligned}
\rho &\equiv \psi^\dagger\psi \\
\mathbf{J}' &\equiv \psi^\dagger\beta_1\boldsymbol{\sigma}\psi
\end{aligned}
\tag{B-1}
$$

A Lorentz boost applied to the wave function has the form:

$$
\psi' = \exp(\beta_1\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2)\psi
\tag{B-2}
$$

Applied to the "probability 4-current":

$$
\begin{aligned}
\rho' &= \psi^{\dagger} \exp(\beta_1 \boldsymbol{\sigma} \cdot \boldsymbol{\alpha})\psi = \psi^{\dagger}[\cosh\alpha + \beta_1 \sigma_\alpha \sinh\alpha]\psi \\
&= \rho \cosh\alpha + J_\alpha \sinh\alpha \\
J'_\alpha &= \psi^{\dagger} \exp(\beta_1 \sigma_\alpha\, \alpha/2)\beta_1 \sigma_\alpha \exp(\beta_1 \sigma_\alpha\, \alpha/2)\psi = \psi^{\dagger}[\beta_1 \sigma_\alpha \cosh\alpha + \sinh\alpha]\psi \\
&= J_\alpha \cosh\alpha + \rho \sinh\alpha \\
\mathbf{J}'_\perp &= \psi^{\dagger} \exp(\beta_1 \sigma_\alpha\, \alpha/2)\beta_1 \boldsymbol{\sigma}_\perp \exp(\beta_1 \sigma_\alpha\, \alpha/2)\psi = \psi^{\dagger}\left[\beta_1 \boldsymbol{\sigma}_\perp \left(\cosh^2\alpha - \sinh^2\alpha\right)\right]\psi \\
&= \mathbf{J}_\perp
\end{aligned}
\tag{B-3}
$$

Notes:

Active Rotation:

$$\partial_{\varphi_j} U_R \psi(\mathbf{r}) = \partial_{\varphi_j} R\psi\left(R^{-1}\mathbf{r}\right) = -\mathrm{i}\frac{\sigma_j}{2}\psi(\mathbf{r}) - \frac{\partial \mathbf{r}}{\partial \varphi_j} \cdot \nabla\psi(\mathbf{r}) = -\mathrm{i}\frac{\sigma_j}{2}\psi(\mathbf{r}) - \mathbf{r} \times \nabla\psi(\mathbf{r})$$

Passive Rotation:

$$\partial_{\varphi_j} \psi(\mathbf{r}) = \partial_{\varphi_j} \psi(\mathbf{r}) = \mathrm{i}\frac{\sigma_j}{2}\psi(\mathbf{r}) + \frac{\partial \mathbf{r}}{\partial \varphi_j} \cdot \nabla\psi(\mathbf{r}) = \mathrm{i}\frac{\sigma_j}{2}\psi(\mathbf{r}) + \mathbf{r} \times \nabla\psi(\mathbf{r})$$

$$\mathbf{w} \times \mathbf{S} = \mathbf{w} \cdot \partial_\varphi\left[\psi^{\dagger}\frac{\boldsymbol{\sigma}}{2}\psi\right] = \frac{1}{2}w_k\psi^{\dagger}\left[\frac{\mathrm{i}\sigma_k\boldsymbol{\sigma}}{2} - \frac{\mathrm{i}\boldsymbol{\sigma}\sigma_k}{2}\right]\psi = \boldsymbol{\sigma}\left[-\frac{1}{2}w_k\psi^{\dagger}\,\mathrm{i}\sigma_k\psi\right]$$

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