Contents

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Chapter 3. Elastic Waves and Quantum Mechanics

"An ocean traveler has even more vividly the impression that the ocean is made of waves than that it is made of water."

— Arthur S. Eddington

3.1. Introduction

"... we must hang on to the basic ideas of logic at all costs."
— Paul Adrian Maurice Dirac [1989]

The theoretical developments discussed in this book were accompanied by myriad experimental discoveries, most notably in the laboratories of J. J. Thomson [Figure 3.1] and his student (and later successor at Cambridge) Ernest Rutherford [Figure 3.2]. J.J. Thomson's study of cathode rays led to his discovery of the electron [1897]. Rutherford [1911, 1914] observed that beams of alpha particles occasionally scatter at large angles from a thin target. This observation led him to propose that atoms contain a positively charged nucleus of extremely small size (of order 10^{-12} cm radius) surrounded by a much larger volume (of order 10^{-8} cm radius) of negatively charged electrons. The Rutherford atomic model became the basis for all future theories of atomic structure.

We have already mentioned the beginnings of quantum theory in the introduction to the previous chapter. Now we will discuss events which led to the development of a wave equation for the electron. This synopsis is based largely on Whittaker [1954].

According to Bohr's atomic model [Bohr 1913] the electron energy levels in hydrogen are:

$$W = -\frac{2\pi^2 e^4 m}{h^2 n^2} = -\frac{e^4 m}{2\hbar^2 n^2} = -R\frac{1}{n^2} \tag{1}$$

where R is called the Rydberg constant. Radiation is emitted when an electron drops from a higher energy level (larger n) to a lower energy level (smaller n), and the frequency of the radiation is proportional to the difference in energies.

William Wilson [1915] and Arnold J. W. Sommerfeld [1915a, 1915b, 1916a] [Figure 3.3] recognized Bohr's quantization of angular momentum of circular orbits (yielding energy quantum number n) to be a special case of quantization of action: $\oint p_i dq_i = h$, where q_i is a coordinate variable and p_i is the corresponding momentum. Sommerfeld explained much of the 'fine structure' of hydrogen spectral lines by generalizing Bohr's circular orbits to ellipses,

including relativistic inertia corrections and a new azimuthal quantum number *k*. The relativistic correction to the energy levels of hydrogen-like atoms is:

$$\Delta W = \frac{e^4}{c^2\hbar^2} R \frac{Z^4}{n^3 k(k-1)} \tag{2}$$

The fine structure constant, $e^2/c\hbar \approx 1/137$, represents the ratio between the velocity of the first Bohr orbit and the speed of light [Whittaker p. 120].

Karl Schwarzchild [1916] and Paus Sophus Epstein [1916] used action quantization to derive the spectral line shifts for hydrogen in a strong electric field (Stark effect). Sommerfeld [1916b] and Peter Debye [1916] explained the splitting of spectral lines in a strong magnetic field (Zeeman effect) by using three quantization conditions: energy (*n*), magnitude of orbital angular momentum ($l = k - 1 \le n$), and component of angular momentum parallel to the applied magnetic field (*m*). Note that $|m| \le l$. Quantization of a single component of angular momentum, termed 'space quantization', was verified when O. Stern and W. Gerlach [1921] split a beam of silver atoms into two discrete components simply by applying a nonuniform magnetic field.

Principal spectral lines of alkali elements (e.g. Na) are doublets which could not be explained by the aforementioned quantum numbers. Various schemes were proposed to include an additional angular momentum quantum number which was generally supposed to be associated with the atomic core. Wolfgang Pauli disputed this identification of core angular momentum in part because it led to a Z^3 dependence in the relativistic energy shifts. He instead attributed the quantum number *j* to the radiant electron which possessed a "classically non-describable two-valuedness". Pauli [1925] also observed that restriction of each set of quantum numbers *n*, *k*, *j*, and *m* to a single electron (the 'exclusion principle') was consistent with the notion of electron shells (proposed by Edmund C. Stoner and J. D. Main Smith) which close when all of the quantum numbers for a given value of *n* are filled by electrons.

Ralph Kronig realized that self-rotation of the electron with angular momentum of $\hbar/2$ would explain the Z^4-dependence of the doublet energy shifts, but since his calculation of the energy levels was off by a factor of two he did not publish his idea. Uhlenbeck and Goudsmidt [1925] did publish the idea of electron angular momentum of $\hbar/2$, but unsuccessfully attempted to withdraw the paper after realizing the factor of two discrepancy. At this time Llewellyn Hilleth Thomas [1926, 1927] resolved the factor of two discrepancy by publishing a paper which demonstrated that the (classical) relativistic precession of the electron magnetic moment in the internal atomic magnetic field, and hence the splitting of energy levels, had been computed incorrectly. Hence the electron's spin angular momentum of $\hbar/2$ was established.

Werner Heisenberg [1925] [Figure 3.4] proposed that transitions between stationary states (e.g. *m* and *n*) could be represented by an array of elements (e.g. x_{mn}) whose amplitude is related to the likelihood of the transition. Max Born [1925] and Pascual Jordan quickly developed this idea into a complete formulation of matrix mechanics in which commutation rules replaced

action integrals as the basis of quantization (e.g. $qp - pq = \mathrm{i}\hbar$ where q is a coordinate and p is the conjugate momentum).

Louis de Broglie [1924] proposed a novel explanation for Bohr's quantization rules. He proposed that matter has a wavelike character with energy proportional to frequency $\varepsilon = \hbar\omega$ and momentum proportional to wave vector $\mathbf{p} = \hbar\mathbf{k}$. The periodic condition for a wave of wavelength λ propagating in a circular orbit of radius r:

$$2\pi r = n\lambda \tag{3}$$

implies quantization of angular momentum:

$$rp = n\hbar \tag{4}$$

Erwin Schrödinger [1926] [Figure 3.5] subsequently published a differential wave equation based on de Broglie's matter waves. For a non-relativistic particle of mass m in a potential $V(\mathbf{r},t)$, the energy is given by:

$$E = \frac{p^2}{2m} + V \tag{5}$$

The corresponding differential equation for de Broglie waves is called the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{6}$$

where the wave function ψ is a complex scalar. For a Coulomb potential ($V = -e^2/r$) this equation yields energy eigenvalues equal to Bohr's energy levels. Schrödinger initially interpreted the wave function to be related to electrical charge density, but Max Born's [1926] interpretation of $\psi^*\psi$ as a probability density was soon widely accepted. A probability conservation equation can be obtained by multiplying ψ^* and adding the complex conjugate:

$$\frac{\partial}{\partial t}|\psi|^2 + -\frac{\mathrm{i}\hbar}{2m}\left\{\psi^*\left[\nabla^2\psi\right] - \left[\nabla^2\psi^*\right]\psi\right\} \tag{7}$$

The Schrödinger equation has the classical Hamiltonian form (see e.g. Goldstein [1980]):

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial t} + H\psi = 0 \tag{8}$$

with $-\mathrm{i}\hbar\psi$ representing Hamilton's principal function whose gradient is the momentum $\mathbf{p}$.

The differential equation corresponding to the relativistic energy-momentum relation $E^2 = p^2c^2 + m_0^2c^4$ is called the Klein-Gordon equation (or relativistic Schrödinger equation):

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi + m_0^2 c^4 \psi \tag{9}$$

Interpretation of this equation proved more difficult than Schrödinger's non-relativistic equation. It does not have the classical Hamiltonian form with a first-order time derivative. The resulting conservation equation is obtained by multiplying ψ^* and subtracting the complex conjugate:

$$\frac{\partial}{\partial t}\left\{\frac{\mathrm{i}\hbar e}{2mc^2}\left[\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right]\right\} + \nabla \cdot \left\{\frac{-\mathrm{i}\hbar e}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]\right\} = 0 \tag{10}$$

The density in this equation (the first square brackets) can have either sign, making it problematic as an expression for probability density. Nonetheless the Klein-Gordon equation eventually became accepted as a description of particles with zero spin.

Schrödinger subsequently demonstrated that Heisenberg's commutation rule $qp - pq = \mathrm{i}\hbar$ follows immediately from the definition of conjugate momenta as derivatives:

$$q\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right)\psi - \left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right) q\psi = \mathrm{i}\hbar\psi \tag{11}$$

Pauli [1927] [Figure 3.6] multiplied Schrodinger's wave function by a two-component factor (termed a *spinor*) to model the two-valued space quantization due to electron spin. Multiplicative operators on Pauli spinors are linear combinations of independent 2×2 matrices which by convention are:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

The last three of these matrices form a vector (i.e. transform as a vector under rotations) and are called the Pauli matrices.

Paul Dirac [1928] [Figure 3.7] finally derived a valid relativistic wave equation by extending the wave function to four components and using matrix coefficients. The Dirac wave function has four complex components which can be written as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \end{bmatrix}^{\mathrm{T}} \tag{13}$$

Such a wave function is called a Dirac spinor or bispinor. A Dirac spinor can be decomposed into left- and right-handed Pauli spinors which each have two complex components. Dirac's equation describing an electron in an electromagnetic potential is:

$$i\hbar\frac{\partial}{\partial t}\psi = -i\hbar c\boldsymbol{\alpha}\cdot\nabla\psi + \beta m_e c^2\psi + \left(e\Phi - e\boldsymbol{\alpha}\cdot\mathbf{A}\right)\psi \tag{14}$$

where $\boldsymbol{\alpha}$ and β are the matrices:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};\quad \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};\quad \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

Dirac also demonstrated that quantum mechanical equations could describe multiple particles by introducing a new wave function whose integrated square magnitude is taken to be the number of particles. This procedure is called "second quantization" (see e.g. [Tomonaga 1974]). Dirac developed this method for bosons by assuming the scalar amplitudes (a_k) of various states (k) to be operators which satisfy the commutation relation $a_k a_l^\dagger - a_k^\dagger a_l = \delta_{kl}$. The product $a_k^\dagger a_k$ then has non-negative integer eigenvalues and represents the number of particles in each state. Jordan and Eugene Wigner [1928] adapted this idea to fermions by using an anti-commutation relation $a_k a_l^\dagger + a_k^\dagger a_l = \delta_{kl}$. In this case the product $a_k^\dagger a_k$ has eigenvalues of zero and one, consistent with Pauli's exclusion principle.

Dirac's equation remains the foundation for describing matter waves. The Standard Model of particle physics "asserts that the material in the universe is made up of elementary fermions interacting through fields, of which they are the sources. The particles associated with the interaction fields are bosons." [Cottingham and Greenwood 1998]. The wave functions are regarded as dimensionless quantities whose magnitude at any point represents a probability density for the presence of one or more particles. Some efforts were made to formulate a classical interpretation of the wave function (notably by de Broglie [1928] and David Bohm [1952], see e.g. Goldstein [2002]) but none was successful in the 20th century.

The mathematical and geometrical properties of spinors were first studied by the mathematician Élie Cartan in 1913 (see e.g. Hladik [1999] for a mathematical analysis of spinors). The algebra of spinors is closely related to that of quaternions, which were invented by Sir William R. Hamilton around 1843 as a generalization of complex numbers to higher dimension. Quaternions consist of four real components. They can in fact be written in matrix form with basis vectors I, σ_x, σ_y, and σ_z.

Spinors have historically been regarded by mathematicians as operators (linear representations of rotation groups) and by physicists as abstract quantities with no classical interpretation. However, David Hestenes [1967] developed a space-time algebra which provides a geometrical interpretation of the Dirac equation. The wave function describes a generalized Lorentz rotation (spatial rotation and velocity boost) in addition to an amplitude and one additional parameter which appears to transform between matter and anti-matter.

There have been successful attempts to reformulate the Dirac theory in terms of relations between local physical observables [Takabayashi 1957, Hestenes 1973]. The Dirac equation uniquely determines the evolution of local dynamical quantities such as angular momentum density, linear momentum density, and energy density. In other words the Dirac equation is deterministic with respect to dynamical quantities.

In this chapter we will derive a Dirac equation to describe rotational waves in an elastic solid. We will regard 'particles' as soliton solutions. We will then derive numerous properties of elementary particles from this model.

3.2. Torsion Waves

...there are circumstances in which mathematics will produce results which no one has really been able to understand in any direct fashion. An example is the Dirac equation, which appears in a very simple and beautiful form, but whose consequences are hard to understand.

— Richard P. Feynman, Robert B. Leighton, and Matthew Sands [1963a]

Quantum theory developed from an initial classical picture of matter as particles. Yet we have seen that special relativity is a natural consequence of the wave nature of matter. Therefore the classical theory which corresponds to quantum mechanics must be a wave theory. One historical dilemma of quantum wave theory is the lack of an obvious physical interpretation of the wave amplitudes. Max Born suggested that the wave intensity be interpreted as a probability density, but he emphasized that "...the probability itself is propagated in accordance with the law of causality" [Born 1926]. While there is no doubt that the quantum wave functions can predict the likelihood of experimental results, their evolution indicates causal rather than stochastic interactions.

Actually, the dynamical interpretation of the wave functions can be resolved by simple dimensional analysis. In terms of Dirac spinors, the *z*-component of spin angular momentum density s_z is:

$$s_z = \left[\frac{\hbar}{\int d^3 r \, |\psi|^2} \right] \frac{1}{2} \psi^\dagger \sigma_z \psi \tag{16}$$

where ψ is the 4-component complex wave function with $|\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ and σ_z is the *z*-component spin angular momentum matrix:

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{17}$$

The leading factor in Eq. 16 is simply a constant which establishes units.

Construction of a classical wave theory of matter must therefore begin with waves carrying angular momentum. Classically, angular momentum is associated with rotations of inertial bodies. Waves of angular momentum require not only inertia but also torque which resists rotations. Generation of torque in response to local rotations implies elasticity. Therefore the classical model of matter waves consists of rotations in an elastic solid (torsion or shear waves). We already know that the elastic solid was the basis for classical wave theories of light, so we can proceed with some confidence.

First consider torsion in one dimension, such as on a torsion wave machine or a stretched-out rubber band (Figure 3.8). A torsion wave machine has at least one intriguing parallel with particle physics. If one rotates a single rod near the center of the wire, a right-handed twist propagates in one direction and a left-handed twist propagates in the other direction, analogous to the production of particles and anti-particles. In every known physical process, anti-matter behaves like a mirror image of matter. Another interesting property of 1-D rotations is that there is a natural distinction between rotations of odd and even multiples of π, analogous to the distinction between odd (fermions) and even (bosons) multiples of the unit angular momentum $\hbar/2$. The notion that torsion should be associated with matter is in fact widely accepted [Kleinert 1989].Therefore there is reason to believe that a mathematical analysis of torsion waves might provide some clues to the interpretation of quantum mechanics. This analogy was first explored by Close [2002].

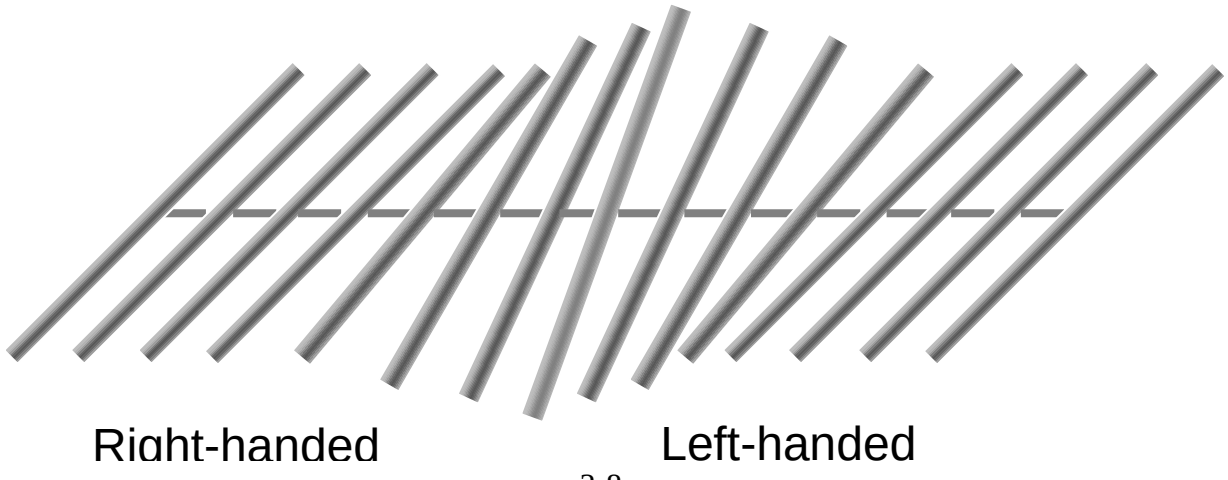

Figure 3.8: Rotation of a single bar on a torsion wave machine results in mirror-symmetric waves propagating in opposite directions. This is a one-dimensional analogue of production of particles and anti-particles. Matter and anti-matter are also typically produced in mirror-symmetric pairs.

If the moment of inertia per unit length is *I*, and the torsion spring constant of the wire (or rubber band) is *K*, then the wave equation is given by:

$$I\frac{\partial^2\Theta(z,t)}{\partial t^2} = K\frac{\partial^2\Theta(z,t)}{\partial z^2} \tag{18}$$

where $\Theta(z,t)$ is the orientation at axial position z and time t. The wave speed is given by $c = \sqrt{K/I}$.

As with displacement waves, a unique frequency and wavelength cannot be defined for torsion waves unless many cycles are produced in succession. If one end of the wave machine is rotated at a constant rate ω, the torsion waves propagate along the machine with uniform wavelength $\lambda = c/\omega$. Each rod along the machine rotates with the constant driving frequency ω. The angular momentum per unit length ℓ is therefore $\ell = I\omega = Ick = Ic\,\partial\Theta/\partial z$. The angular momentum is therefore proportional to the spatial derivative of the angle. The angular momentum of a twist from 0 to Θ_0 can be obtained by integrating over angle:

$$L_0 = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} I\frac{d\Theta}{dt}dz = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} Ic\frac{d\Theta}{dz}dz = \int_0^{\Theta_0} Icd\Theta = Ic\Theta_0 \tag{19}$$

Thus we see that the total angular momentum of a twist is proportional to the rotation angle and independent of frequency.

A twist propagating with constant wavelength has no torque, so the kinetic and potential energies remain constant as the wave propagates. The kinetic energy per unit length is $I\omega^2/2$ and the potential energy per unit length is $\frac{1}{2}K(\partial\Theta/\partial z)^2 = 2\pi^2K/\lambda^2 = I\omega^2/2$. Integration from 0 to Θ_0 yields for the total energy:

$$\varepsilon = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} I\omega\left|\frac{d\Theta}{dt}\right|dz = \int_{z(\Theta=0)}^{z(\Theta=\Theta_0)} Ic\omega\left|\frac{d\Theta}{dz}\right|dz = \int_0^{\Theta_0} Ic\omega d\Theta = Ic\omega\Theta_0 = L_0\omega \tag{20}$$

The wave energy is equal to the wave angular momentum times the angular frequency. This is analogous to the energy quantum of $\hbar\omega$. At this point we make the identifications:

$$I = \frac{L_0}{c\Theta_0}$$
$$K = c^2 I = \frac{L_0 c}{\Theta_0} \tag{21}$$

so that the wave equation is simply:

$$\frac{\partial^2 \Theta(z,t)}{\partial t^2} = c^2 \frac{\partial^2 \Theta(z,t)}{\partial z^2} \tag{22}$$

Incidentally, although we have been describing torsion waves along a thin wire, the equation is valid for torsion waves in a thick cylindrical rod (see e.g. Feynman et al. 1963b).

Now we will take a look at the classical wave equation to see if it can be applied to the study of matter. We will start with one-dimensional waves as above, then generalize to three dimensional scalar and vector waves.

3.3. One-Dimensional Scalar Waves

"I have deep faith that the principle of the universe will be beautiful and simple."

—Albert Einstein

Consider a scalar quantity (a) which satisfies a wave equation with wave speed (c) in one spatial dimension (z):

$$\partial_t^2 a = c^2 \partial_z^2 a \tag{23}$$

This equation can be factored:

$$[\partial_t + c\partial_z][\partial_t - c\partial_z]a = 0 \tag{24}$$

The general solution is a superposition of forward (a_F) and backward (a_B) propagating waves:

$$a(z,t) = a_F(z-ct) + a_B(z+ct) \tag{25}$$

This form of the solution to the one-dimensional wave equation can be found in any elementary textbook on waves. We can write the equations for forward and backward waves in matrix form:

$$\left[\partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c\partial_z\right]\begin{pmatrix} a_F(z-ct) \\ a_B(z+ct) \end{pmatrix} = 0 \tag{26}$$

The spatial derivatives are related to the temporal derivatives:

$$c\partial_z\begin{pmatrix} a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct)\end{pmatrix} = \partial_t\begin{pmatrix} -a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct)\end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\partial_t\begin{pmatrix} a_{\mathrm{F}}(z-ct) \\ a_{\mathrm{B}}(z+ct)\end{pmatrix} \tag{27}$$

Let $\dot{a} \equiv \partial_t a$ and $a' \equiv \partial_z a$. We now define a wave function in terms of the time derivatives:

$$\Psi \equiv \begin{pmatrix} \dot{a}_{\mathrm{F}}(z-ct) \\ \dot{a}_{\mathrm{B}}(z+ct)\end{pmatrix} \tag{28}$$

The wave equation for the forward and backward waves is now:

$$\left[\partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} c\partial_z\right]\Psi = \partial_t\begin{pmatrix} \dot{a}_{\mathrm{F}}(z-ct) \\ \dot{a}_{\mathrm{B}}(z+ct)\end{pmatrix} - c\partial_z\begin{pmatrix} a'_F(z-ct) \\ a'_B(z+ct)\end{pmatrix} = 0 \tag{29}$$

We have now reduced the second-order wave equation to a first-order matrix equation.

3.3.1. Spinors and Bispinors

If we regard the *z*-axis as one of three orthogonal axes, then the two independent components $\dot{a}_{\mathrm{F}}$ and $\dot{a}_{\mathrm{B}}$ differ by a 180 degree rotation. This is the definitive property of independent states in spin one-half systems. Unfortunately, this property is de-emphasized (or even unrecognized) in the physics literature in favor of the more exotic property that complex spinors change sign upon 360 degree rotation. This latter property does not apply to physical observables which are computed from bilinear products of spinors. However, the separation of independent states by 180 degrees does apply to wave velocity, implying that solutions of the wave equation generally form spin one-half systems. Note that unlike positive and negative scalars or vector components (which can also be expressed as bilinear products of spinors), waves with positive and negative velocity are not related by a multiplicative factor of minus one. The forward and backward waves are independent states. The mathematical basis of this property is that wave velocity is a property of the functional arguments and is not simply an amplitude.

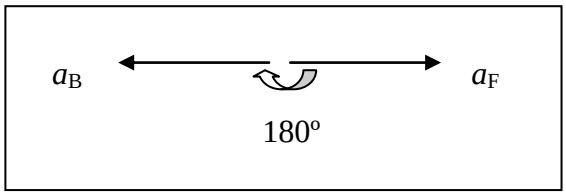

Figure: Waves propagating in opposite directions along an axis comprise independent states separated by a 180 degree rotation. This is the basis of half-integer spin.

The relationship between waves and spinors can be made explicit as in Close (2002) by further decomposition into positive-definite components ($\dot{a}_{\mathrm{F}+}, \dot{a}_{\mathrm{B}+}, \dot{a}_{\mathrm{F}-}, \dot{a}_{\mathrm{B}-}$) or ($a'_{\mathrm{F}+}, a'_{\mathrm{B}+}, a'_{\mathrm{F}-}, a'_{\mathrm{B}-}$) representing positive (+) or negative (–) contributions to the wave derivatives:

$$\dot{a}(z,t) = \dot{a}_{\mathrm{F}+}(z-ct) - \dot{a}_{\mathrm{F}-}(z-ct) + \dot{a}_{\mathrm{B}+}(z+ct) - \dot{a}_{\mathrm{B}-}(z+ct) \tag{30}$$

and

$$\begin{aligned} ca'(z,t) &= c\left[a'_{\mathrm{F}+}(z-ct) - a'_{\mathrm{F}-}(z-ct) + a'_{\mathrm{B}+}(z+ct) - a'_{\mathrm{B}-}(z+ct)\right] \\ &= -\dot{a}_{\mathrm{F}+}(z-ct) + \dot{a}_{\mathrm{F}-}(z-ct) + \dot{a}_{\mathrm{B}+}(z+ct) - \dot{a}_{\mathrm{B}-}(z+ct) \end{aligned} \tag{31}$$

From here on the functional arguments will not be written explicitly. Note that the positive-definite components may have discontinuous derivatives where the original signed quantities pass continuously through zero. For example, to make the time derivatives continuous requires matching conditions for $\dot{a}$:

$$\begin{aligned} \partial_t \dot{a}_{\mathrm{F}+}\big|_{\dot{a}_{\mathrm{F}+}=\dot{a}_{\mathrm{F}-}=0} &= -\partial_t \dot{a}_{\mathrm{F}-}\big|_{\dot{a}_{\mathrm{F}+}=\dot{a}_{\mathrm{F}-}=0} \\ \partial_z \dot{a}_{\mathrm{F}+}\big|_{\dot{a}_{\mathrm{F}+}=\dot{a}_{\mathrm{F}-}=0} &= -\partial_z \dot{a}_{\mathrm{F}-}\big|_{\dot{a}_{\mathrm{F}+}=\dot{a}_{\mathrm{F}-}=0} \end{aligned} \tag{32}$$

Similar relations hold for the backward wave components. Such discontinuities do not affect the validity of the first order equations. However, higher derivatives may be undefined at some points.

Since each component has a unique sign, we can express $\dot{a}$ and a' in spinorial form with the one-dimensional wave function ψ_{v} (the subscript 'v' refers to the velocity axis):

$$\dot{a} = \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv \psi_{\mathrm{v}}^T \sigma \psi_{\mathrm{v}}$$

$$ca' = \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv -\psi_{\mathrm{v}}^T \beta \psi_{\mathrm{v}} \tag{33}$$

where the superscript *T* indicates transposition of the column matrix and the matrix $\beta\sigma$ tabulates the forward and backward velocities (v):

$$v\psi_{\mathrm{v}} = c\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \\ \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \end{pmatrix} \equiv c\beta\sigma\psi_{\mathrm{v}} \tag{34}$$

This wave function is a one-dimensional bispinor. In one dimension the components of the bispinor may be taken to be real and positive-definite. Extension to three dimensions requires complex components.

Changing the order of terms in the wave function is called a change of 'representation'. A few important points are:

1. The components of the column matrix wave function are real and positive-definite.

2. Only one forward component and one backward component can be non-zero at any given time and place (for one-dimensional waves).

3. The spatio-temporal variation of each component must be consistent with its location in the column matrix.

Since some of the components must be zero, let δ_{F} and δ_{B} be either zero or one. Then the wave function is:

$$\psi_{\mathrm{v}} = \left[\dot{a}_{\mathrm{F}}^{1/2}\delta_{\mathrm{F}} \quad \dot{a}_{\mathrm{B}}^{1/2}\delta_{B} \quad \dot{a}_{\mathrm{B}}^{1/2}\left[1-\delta_{\mathrm{B}}\right] \quad \dot{a}_{\mathrm{F}}^{1/2}\left[1-\delta_{\mathrm{F}}\right]\right]^{\mathrm{T}} \tag{35}$$

Using Lorentz boosts, the wave function can be written as:

$$\psi_{\mathrm{v}} = \dot{a}_{0}^{1/2}\exp\left(\beta\sigma\,\alpha/2\right)\left[\delta_{\mathrm{F}} \quad \delta_{B} \quad \left[1-\delta_{B}\right] \quad \left[1-\delta_{\mathrm{F}}\right]\right]^{\mathrm{T}}/\sqrt{2} \tag{36}$$

This form has two independent continuous parameters and two binary parameters.
The equation of evolution of the wave components is:

$$\partial_{t}\psi_{\mathrm{v}} + c\beta\sigma\,\partial_{z}\psi_{\mathrm{v}} = 0 \tag{37}$$

This is the one-dimensional Dirac equation. This equation can be interpreted as a convective derivative with two opposite velocities represented by the matrix $v=c\beta\sigma$.

The relation between one dimensional bispinor equations and scalar wave equations is summarized in Table 1.

Table 1. Corresponding Bispinor and Scalar Wave Equations in One Dimension

Bispinor Equation	Scalar Equation
$\partial_t\left[\psi_{\mathrm{v}}^{\mathrm{T}}\sigma\psi_{\mathrm{v}}\right]+c\partial_z\left[\psi_{\mathrm{v}}^{\mathrm{T}}\beta\psi_{\mathrm{v}}\right]=0$	$\partial_t^2 a-c^2\partial_z^2 a=0$
$\partial_t\left[\psi_{\mathrm{v}}^{\mathrm{T}}\psi_{\mathrm{v}}\right]+c\partial_z\left[\psi_{\mathrm{v}}^{\mathrm{T}}\beta\sigma\psi_{\mathrm{v}}\right]=0$	$\partial_t\lvert\partial_t a_F\rvert+\partial_t\lvert\partial_t a_B\rvert+c^2\partial_z\lvert\partial_z a_F\rvert-c^2\partial_z\lvert\partial_z a_B\rvert=0$
$\partial_t\left[\psi_{\mathrm{v}}^{\mathrm{T}}\beta\psi_{\mathrm{v}}\right]+c\partial_z\left[\psi_{\mathrm{v}}^{\mathrm{T}}\sigma\psi_{\mathrm{v}}\right]=0$	$\partial_t\left[-c\partial_z a\right]+c\partial_z\left[\partial_t a\right]=0$
$\partial_t\left[\psi_{\mathrm{v}}^{\mathrm{T}}\beta\sigma\psi_{\mathrm{v}}\right]+c\partial_z\left[\psi_{\mathrm{v}}^{\mathrm{T}}\psi_{\mathrm{v}}\right]=0$	$c\partial_t\lvert\partial_z a_F\rvert-c\partial_t\lvert\partial_z a_B\rvert+c\partial_z\lvert\partial_t a_F\rvert+c\partial_z\lvert\partial_t a_B\rvert=0$

3.3.2. Wave Velocity

The mean velocity (v) of the wave is proportional to the ratio between the difference and sum of the forward and backward components [Close 2002]:

$$v=c\frac{\lvert\dot{a}_F\rvert-\lvert\dot{a}_B\rvert}{\lvert\dot{a}_F\rvert+\lvert\dot{a}_B\rvert}=c\frac{\psi_{\mathrm{v}}^T\beta\sigma\psi_{\mathrm{v}}}{\psi_{\mathrm{v}}^T\psi_{\mathrm{v}}} \tag{38}$$

Since $\lvert\dot{a}_F\rvert$ and $\lvert\dot{a}_B\rvert$ are positive-definite, we can define them by the relation:

$$\begin{aligned}\lvert\dot{a}_F\rvert&=\dot{a}_0\exp(\alpha)\\ \lvert\dot{a}_B\rvert&=\dot{a}_0\exp(-\alpha)\end{aligned} \tag{39}$$

so that our definition of velocity is:

$$v=c\frac{\dot{a}_0\exp(\alpha)-\dot{a}_0\exp(-\alpha)}{\dot{a}_0\exp(\alpha)+\dot{a}_0\exp(-\alpha)}=c\tanh\alpha \tag{40}$$

If we start from a zero-velocity state with $\lvert\dot{a}_{\mathrm{F}}\rvert=\lvert\dot{a}_{\mathrm{B}}\rvert=\dot{a}_0$, then we can change the velocity using the 'Lorentz boost' operator ($\psi_{\mathrm{v}}\rightarrow\exp(\beta\sigma\alpha/2)\psi_{\mathrm{v}}$):

$$v=\frac{\left[\psi_{\mathrm{v}}^T\exp(\beta\sigma\alpha/2)\right]c\beta\sigma\left[\exp(\beta\sigma\alpha/2)\psi_{\mathrm{v}}\right]}{\left[\psi_{\mathrm{v}}^T\exp(\beta\sigma\alpha/2)\right]\left[\exp(\beta\sigma\alpha/2)\psi_{\mathrm{v}}\right]}=c\frac{\exp(\alpha)-\exp(-\alpha)}{\exp(\alpha)+\exp(-\alpha)}=c\tanh\alpha \tag{41}$$

Note that successive boosts preserve the form of the operator:

$$\exp(\beta\sigma\alpha_2/2)\exp(\beta\sigma\alpha_1/2)=\exp(\beta\sigma\left[\alpha_1+\alpha_2\right]/2) \tag{42}$$

This property enables us to recover the relativistic equation for addition of parallel velocities:

$$\begin{aligned} v_1 &= c\tanh\alpha_1 \\ v_2 &= c\tanh\alpha_2 \\ v &= c\tanh(\alpha_1+\alpha_2) = c\frac{\tanh\alpha_1+\tanh\alpha_2}{1+\tanh\alpha_1\tanh\alpha_2} = \frac{v_1+v_2}{1+v_1v_2/c^2} \end{aligned} \tag{43}$$

This result is another example of how the laws of special relativity apply to classical waves in ordinary Galilean space-time, as discussed in Chapter 2.

Using Lorentz boosts, the wave function can be written as:

$$\psi_{\mathrm{v}} = \dot{a}_0^{1/2}\exp(\beta\sigma\,\alpha/2)\left[\delta_{\mathrm{F}} \quad \delta_B \quad [1-\delta_{\mathrm{F}}] \quad [1-\delta_B]\right]^T/\sqrt{2} \tag{44}$$

This form has two independent continuous parameters and two binary parameters.

3.4. Three Dimensional Scalar Waves

"... in quantum phenomena one obtains quantum numbers, which are rarely found in mechanics but occur very frequently in wave phenomena and in all problems dealing with wave motion."

— Louis de Broglie [1963]

3.4.1. Rotation of Gradient and Velocity

The spatial derivative ∂_z generalizes in three dimensions to a arbitrary direction ∂_{v}, where the index (v) represents an arbitrary direction. Wave velocity is defined to be parallel to the gradient. Since the matrix $\beta\sigma$ is associated with a particular axis, it must be one component of a vector. We can let the matrix $\beta \equiv \beta_3$ and define the gradient matrix components as:

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & -\tilde{\mathrm{i}} & 0 \\ 0 & 0 & 0 & -\tilde{\mathrm{i}} \\ \tilde{\mathrm{i}} & 0 & 0 & 0 \\ 0 & \tilde{\mathrm{i}} & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{45}$$

The symbol ($\tilde{\mathrm{i}}$) represents a unit pseudoscalar imaginary which is odd (changes sign) with respect to spatial inversion. This property is necessary because velocity is a polar vector and:

$$\tilde{\mathrm{i}} = \beta_1\beta_2\beta_3 \tag{46}$$

We must now allow the wave function to have complex components. These matrices have *commutation relations* equivalent to the Pauli matrices:

$$\beta_i\beta_j + \beta_j\beta_i = 2\delta_{ij} \;;\;\; \beta_i\beta_j - \beta_j\beta_i = 2\,\tilde{\mathrm{i}}\,\varepsilon_{ijk}\beta_k \tag{47}$$

An elegant way to write these commutation relations is:

$$\beta_i\beta_j = \beta_i \cdot \beta_j + \tilde{\mathrm{i}}\,\beta_i \times \beta_j \tag{48}$$

where:

$$\begin{aligned} \beta_i \cdot \beta_j &= \frac{1}{2}\left[\beta_i\beta_j + \beta_j\beta_i\right] \\ \beta_i \times \beta_j &= -\frac{\tilde{\mathrm{i}}}{2}\left[\beta_i\beta_j - \beta_j\beta_i\right] \end{aligned} \tag{49}$$

Hence we can regard these matrices as basis vectors whose commutation relations express their relative orientation. This idea is the basis for the mathematical field of *geometric algebra*. Notice that the unit imaginary now has a geometrical interpretation as the product of three orthogonal unit vectors:

$$\beta_1\beta_2\beta_3 = \beta_1\left[\beta_2 \cdot \beta_3 + \tilde{\mathrm{i}}\,\beta_2 \times \beta_3\right] = \tilde{\mathrm{i}}\,\beta_1 \cdot \left[\beta_2 \times \beta_3\right] = \tilde{\mathrm{i}} \tag{50}$$

The rotation operators for this space have the form:

$$R_{\beta_j}(\zeta_i) = \exp\left(-\tilde{\mathrm{i}}\,\beta_i\zeta_i/2\right)\beta_j \exp\left(\tilde{\mathrm{i}}\,\beta_i\zeta_i/2\right) = \beta_j\cos\zeta_i - \frac{\tilde{\mathrm{i}}}{2}\left[\beta_i\beta_j - \beta_j\beta_i\right]\sin\zeta_i \tag{51}$$

which can be written in vector form:

$$R_{\boldsymbol{\sigma}}(\boldsymbol{\zeta}) = \exp\left(-\tilde{\mathrm{i}}\,\beta_\zeta\zeta/2\right)\boldsymbol{\beta}\exp\left(\tilde{\mathrm{i}}\,\beta_\zeta\zeta/2\right) = \boldsymbol{\beta}\cos\zeta + \left[\boldsymbol{\beta}_\zeta \times \boldsymbol{\beta}\right]\sin\zeta_i \tag{52}$$

To include rotations, the one-dimensional derivative $c\partial_{\mathrm{v}}a = -\psi_{\mathrm{v}}^T\beta_{\mathrm{v}}\psi_{\mathrm{v}}$ must be modified to include orientation. This orientation is computed relative to the x_3-axis. Using the definitions:

$$\begin{aligned} \beta_{\mathrm{v}} &= \left[\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\beta_3\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\right] \\ \psi &= \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_{\mathrm{v}} \\ \psi^\dagger &= \psi_{\mathrm{v}}^T\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right) \end{aligned} \tag{53}$$

The wave function now has complex components. The rotation operator $R_\zeta(\psi_{\mathrm{v}}) = \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\psi_{\mathrm{v}}$ applied to the one-dimensional wave function inverts the rotation of

the basis vectors so that the derivative can be evaluated using the one-dimensional real-valued matrix β_3 and wave function ψ_v.

The spatial derivative is:

$$c\partial_v a = -\psi_v^T \beta_3 \psi_v = -\psi^\dagger \beta_v \psi \tag{54}$$

Since the beta matrices are mutually orthogonal, the components of $\psi^\dagger \boldsymbol{\beta} \psi$ perpendicular to x_v must be zero. Therefore the three dimensional gradient is:

$$c\nabla a = -\hat{e}_v \psi_v^T \beta_3 \psi_v = -\psi^\dagger \boldsymbol{\beta} \psi \tag{55}$$

3.4.2. Successive Rotations

Successive rotations can be performed using either fixed axes or embedded axes. The result of successive rotations about fixed axes depends on the order in which the rotations are taken. For example, successive rotations of $\pi/2$ about the x_1- and x_2-axes move e_3 to either $-e_2$ or $+e_1$, depending on the order. Hence:

$$\begin{aligned} &\exp\left(-\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\left[\exp\left(-\tilde{\mathrm{i}}\,\beta_1\pi/4\right)\beta_3\exp\left(\tilde{\mathrm{i}}\,\beta_1\pi/4/2\right)\right]\exp\left(\tilde{\mathrm{i}}\,\beta_2\pi/4\right) = -\beta_2 \\ &\exp\left(-\tilde{\mathrm{i}}\,\beta_1\pi/4\right)\left[\exp\left(-\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\beta_3\exp\left(\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\right]\exp\left(\tilde{\mathrm{i}}\,\beta_1\pi/4/2\right) = \beta_1 \end{aligned} \tag{56}$$

Here the expression inside the square brackets is evaluated first, followed by applying the rotation operator outside the square brackets. If we interpret these rotation operators as acting on spinors then the order appears to be backward. The expression: $\exp\left(\tilde{\mathrm{i}}\,\beta_1\pi/4/2\right)\exp\left(\tilde{\mathrm{i}}\,\beta_2\pi/4\right)\psi$ represents spinor rotation of $-\pi/2$ about the x_1-axis followed by rotation about the x_2-axis.

3.4.2.1. Euler Angles

We can put the operations back in order if we consider the second rotation operator to have been rotated along with the wave function by the first one:

$$R'(\boldsymbol{\Theta}_2) = R(\boldsymbol{\Theta}_1)R(\boldsymbol{\Theta}_2)R^{-1}(\boldsymbol{\Theta}_1) = \exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\Theta}_1/2\right)\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\Theta}_2/2\right)\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\Theta}_1/2\right) \tag{57}$$

Two successive rotations yields:

$$R'(\boldsymbol{\Theta}_2)R(\boldsymbol{\Theta}_1) = \left[R(\boldsymbol{\Theta}_1)R(\boldsymbol{\Theta}_2)R^{-1}(\boldsymbol{\Theta}_1)\right]R(\boldsymbol{\Theta}_1) = \exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\Theta}_1/2\right)\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\Theta}_2/2\right) \tag{58}$$

Axes which are rotated along with the spinors are called embedded axes. Rotation angles which refer to embedded axes are called *Euler angles*. We use primes to denote rotations about embedded axes.

The Euler rotation operator $R'(\boldsymbol{\Theta}_2)$ can be interpreted as follows: First, rotate the spinor back to its original orientation. Next, rotate the spinor about the fixed axis corresponding to $\boldsymbol{\Theta}_2$. Finally, rotate again about the embedded axis corresponding to $\boldsymbol{\Theta}_1$ (the original axis now rotated by $\boldsymbol{\Theta}_2$). The equation states that rotation by $\boldsymbol{\Theta}_1=\boldsymbol{\Theta}'_1$ followed by rotation about the fixed axis $\boldsymbol{\Theta}_2$ is equivalent to rotation first by $\boldsymbol{\Theta}_2$ followed by rotation by $\boldsymbol{\Theta}'_1$ about the embedded $\hat{\boldsymbol{\Theta}}'_1$ axis. In the above example, rotation by $\pi/2$ about x followed by $\pi/2$ about z (or y') is equivalent to rotation by $\pi/2$ about z followed by $\pi/2$ about y (or x').

The angular derivative of the wave function is:

$$\partial_{\boldsymbol{\varphi}}\psi = \partial_{\boldsymbol{\varphi}}\left[\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\psi_{\mathrm{v}}\right] = -\tilde{\mathrm{i}}\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\frac{\boldsymbol{\beta}}{2}\psi_{\mathrm{v}} = -\tilde{\mathrm{i}}\left[\exp\left(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\frac{\boldsymbol{\beta}}{2}\exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\varphi}/2\right)\right]\psi \tag{59}$$

It is customary in quantum mechanics to define the angular derivative to be:

$$\partial_{\boldsymbol{\varphi}'}\psi = -\tilde{\mathrm{i}}\frac{\boldsymbol{\beta}}{2}\psi \tag{60}$$

This relation is only valid if the angle $\boldsymbol{\varphi}'$ is measured with respect to the embedded axes.

Accumulated rotations can be computed from successive rotations about embedded axes. Given a rotation rate $\mathbf{w}'(t)$ with respect to embedded axes, the accumulated rotation operator is:

$$R(t) = R(\boldsymbol{\Theta}'(t)) = \exp\left(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\Theta}'/2\right) = \exp\left(\tilde{\mathrm{i}}\int dt\,\boldsymbol{\beta}\cdot\mathbf{w}'/2\right) \tag{61}$$

3.4.2.2. Examples

Let us verify this expression with explicit examples. First, we compute the general expression for rotation about two successive embedded axes: Rotate by angle $\boldsymbol{\theta}'_a$ about an axis x'_a followed by $\boldsymbol{\theta}'_b$ about x'_b. The rotation operator is:

$$\begin{aligned} R(t) &= R(\boldsymbol{\Theta}'(t)) = \exp\left(\tilde{\mathrm{i}}\,\beta_b\theta'_b/2\right)\exp\left(\tilde{\mathrm{i}}\,\beta_a\theta'_a/2\right) \\ &= \left[\cos\frac{\theta'_b}{2} + \tilde{\mathrm{i}}\,\beta_b\sin\frac{\theta'_b}{2}\right]\left[\cos\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\,\beta_a\sin\frac{\theta'_a}{2}\right] \\ &= \cos\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2} - \beta_b\beta_a\sin\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\left[\beta_a\cos\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \beta_b\sin\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2}\right] \end{aligned} \tag{62}$$

Recall that $\beta_a\beta_b = \beta_a\cdot\beta_b + \tilde{\mathrm{i}}\,\beta_a\times\beta_b$. We consider two cases. First, if $\boldsymbol{\theta}'_a$ and $\boldsymbol{\theta}'_b$ are parallel then:

$$\begin{aligned} R(\boldsymbol{\Theta}'(t)) &= \cos\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2} - \sin\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \mathrm{i}\,\beta_a\left[\cos\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \sin\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2}\right] \\ &= \cos\left(\frac{\theta'_b + \theta'_a}{2}\right) + \mathrm{i}\,\beta_a \sin\left(\frac{\theta'_b + \theta'_a}{2}\right) \end{aligned} \tag{63}$$

which is obviously correct since parallel angles are additive. Next consider two perpendicular axes with $\beta_b\beta_a = \tilde{\mathrm{i}}\,\beta_c$:

$$R(\boldsymbol{\Theta}'(t)) = \cos\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\,\beta_c \sin\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \tilde{\mathrm{i}}\left[\beta_a \cos\frac{\theta'_b}{2}\sin\frac{\theta'_a}{2} + \beta_b \sin\frac{\theta'_b}{2}\cos\frac{\theta'_a}{2}\right] \tag{64}$$

For the special case where both angles are $\pi/2$ this yields:

$$\begin{aligned} R(\boldsymbol{\Theta}'(t)) &= \cos^2\frac{\pi}{4} + \tilde{\mathrm{i}}\,\beta_c \sin^2\frac{\pi}{4} + \tilde{\mathrm{i}}\left[\beta_a \cos\frac{\pi}{4}\sin\frac{\pi}{4} + \beta_b \sin\frac{\pi}{4}\cos\frac{\pi}{4}\right] \\ &= \frac{1}{2} + \tilde{\mathrm{i}}\,\frac{\beta_a + \beta_b + \beta_c}{2} = \cos(\pi/3) + \tilde{\mathrm{i}}\,\frac{\beta_a + \beta_b + \beta_c}{\sqrt{3}}\sin(\pi/3) \end{aligned} \tag{65}$$

This corresponds to a rotation operator for $2\pi/3$ radians about the axis $[\hat{\mathbf{x}}_a + \hat{\mathbf{x}}_b + \hat{\mathbf{x}}_c]/\sqrt{3}$. The validity of this result can be verified by picturing an equilateral triangle with corners on each axis equidistant from the origin. Clearly rotation by $2\pi/3$ about the center of the triangle merely permutes the positions of the axes, which is of course what happens when rotating by $\pi/2$ around successive orthogonal axes. Note also that the symmetry of the final result implies that:

$$\exp(\tilde{\mathrm{i}}\,\beta_y\,\pi/4)\exp(\tilde{\mathrm{i}}\,\beta_x\,\pi/4) = \exp(\tilde{\mathrm{i}}\,\beta_z\,\pi/4)\exp(\tilde{\mathrm{i}}\,\beta_y\,\pi/4) = \exp(\tilde{\mathrm{i}}\,\beta_x\,\pi/4)\exp(\tilde{\mathrm{i}}\,\beta_z\,\pi/4) \tag{66}$$

(x followed by y', y followed by z', z followed by x') which is consistent with our explanation of the secondary rotation operator above.

3.4.3. Wave Function

In three dimensions the gradient can be defined as a one-dimensional derivative rotated by angle ζ to a new axis $\hat{\mathbf{v}}$. Let:

$$\begin{aligned} \beta_{\mathrm{v}} &\equiv \exp(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\zeta/2)\beta_3 \exp(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\zeta/2) \\ \psi &\equiv \exp(-\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\zeta/2)\psi_{\mathrm{v}} \end{aligned} \tag{67}$$

Rotation by angle ζ is denoted R_ζ and defined relative to a default orientation along the x_3 axis. The three-dimensional gradient is:

$$\nabla a = R_\zeta \left\{ \hat{\mathbf{x}}_3 \frac{\partial a}{\partial x_3} \right\} = -\hat{\mathbf{v}} \psi_{\mathrm{v}}^T c \beta_3 \psi_{\mathrm{v}} = -c \psi^\dagger \boldsymbol{\beta} \psi \tag{68}$$

Writing a column matrix as the transpose of a row matrix, the rotated wave function ψ is:

$$\psi = \dot{a}_0^{1/2} \exp\left(-\tilde{\mathrm{i}} \boldsymbol{\beta} \cdot \boldsymbol{\zeta}/2\right) \exp\left(\beta_3 \sigma \alpha/2\right) \left[\delta_{\mathrm{F}} \quad \delta_B \quad [1-\delta_{\mathrm{F}}] \quad [1-\delta_B]\right]^T / \sqrt{2} \tag{69}$$

However, in three dimensions the constant column matrix which represents v_3=0 states may have nonzero velocity perpendicular to x_3. This is indeed the case for $\psi_0 = \left[1 \quad 0 \quad 1 \quad 0\right]^T / \sqrt{2}$ and $\psi_0 = \left[0 \quad 1 \quad 0 \quad 1\right]^T / \sqrt{2}$. The remaining states with zero velocity are obtained by rotation of velocity from:

$$\psi_0 = \left[1 \quad 0 \quad 0 \quad 1\right]^T / \sqrt{2} \tag{70}$$

This state has zero time derivative but nonzero gradient. When Lorentz boosts are applied both the time derivative and velocity can be non-zero. The final form of the wave function is thus:

$$\psi = \dot{a}_0^{1/2} \exp\left(-\tilde{\mathrm{i}} \boldsymbol{\beta} \cdot \boldsymbol{\zeta}/2\right) \exp\left(\beta_3 \sigma \alpha/2\right) \psi_0 \tag{71}$$

This is the general form of the scalar wave function. The constant matrix is multiplied by factors representing an amplitude, a 1-D velocity boost, and a general rotation in velocity space (two angles to determine velocity direction plus rotation about the velocity axis). Clearly four parameters are needed to determine $\partial_t a$ and ∇a. The significance of rotation about the velocity axis will be discussed below.

3.4.4. First-Order Wave Equation

The time derivative of (71) yields the first-order equation:

$$\partial_t \psi = -\exp\left(-\tilde{\mathrm{i}} \boldsymbol{\beta} \cdot \boldsymbol{\zeta}/2\right) \partial_t \boldsymbol{\zeta} \cdot \frac{\tilde{\mathrm{i}} \boldsymbol{\beta}}{2} \psi_{\mathrm{v}} + \exp\left(-\tilde{\mathrm{i}} \boldsymbol{\beta} \cdot \boldsymbol{\zeta}/2\right) \partial_t \psi_{\mathrm{v}} \tag{72}$$

Here we can see the effect of rotation about the velocity axis. Rotation of the left-hand side involves only direct rotation of the wave function, but rotation of the right-hand side also involves rotation of the angular frequency $\partial_t \boldsymbol{\zeta}$. Rotation about the velocity (or gradient) axis can change the direction of this angular frequency. This is the significance of the fifth parameter in the factorization above.

Inverting the rotation factor yields the one-dimensional wave function, which satisfies the one-dimensional wave equation:

$$\left[\partial_t + c \beta_3 \sigma \hat{\mathbf{v}} \cdot \nabla\right] \left[\exp\left(\tilde{\mathrm{i}} \boldsymbol{\beta} \cdot \boldsymbol{\zeta}/2\right) \psi\right] = 0 \tag{73}$$

Derivatives of the exponential factors are:

$$\begin{aligned}\partial_t \exp(\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2) &= \frac{\tilde{\mathrm{i}}}{2}\exp(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2)\partial_t[\boldsymbol{\beta}\cdot\boldsymbol{\zeta}] \\ \nabla \exp(\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2) &= \frac{\tilde{\mathrm{i}}}{2}\exp(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2)\nabla[\boldsymbol{\beta}\cdot\boldsymbol{\zeta}]\end{aligned} \tag{74}$$

Substituting $\sigma\beta_3 \exp(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2)\hat{\mathbf{v}}\cdot\nabla = \exp(\tilde{\mathrm{i}}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2)\sigma\boldsymbol{\beta}\cdot\nabla$ into (73) yields:

$$\partial_t\psi + c\sigma\boldsymbol{\beta}\cdot\nabla\psi = -\partial_t\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\sigma\boldsymbol{\beta}\cdot\nabla\left[\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\right]\psi \tag{75}$$

This equation states that the convective derivative is nonzero only due to (convective) rotation of velocity direction.

The equation of evolution of the scalar wave amplitude is obtained by multiplying $\psi^\dagger\sigma$ and adding the adjoint:

$$\begin{aligned}\partial_t\left[\psi^\dagger\sigma\psi\right] &= \left\{-c\sigma\boldsymbol{\beta}\cdot\nabla\psi - \frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\sigma\boldsymbol{\beta}\cdot\nabla\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi\right\}^\dagger\sigma\psi \\ &\quad + \psi^\dagger\sigma\left\{-c\sigma\boldsymbol{\beta}\cdot\nabla\psi - \frac{\partial\boldsymbol{\zeta}}{\partial t}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi - c\sigma\boldsymbol{\beta}\cdot\nabla\boldsymbol{\zeta}\cdot\frac{\tilde{\mathrm{i}}\,\boldsymbol{\beta}}{2}\psi\right\} \\ &= -\nabla\cdot\psi^\dagger c\boldsymbol{\beta}\psi + c[\nabla\times\boldsymbol{\zeta}]\cdot\left[\psi^\dagger\boldsymbol{\beta}\psi\right]\end{aligned} \tag{76}$$

Which, in terms of the scalar polarization is:

$$\partial_t^2 a = c^2\nabla^2 a - c^2[\nabla\times\boldsymbol{\zeta}]\cdot\nabla a \tag{77}$$

The relations between rotation angles and velocity unit vectors are:

$$\begin{aligned}&\partial_t\boldsymbol{\zeta} = \hat{e}_\mathrm{v}\times\partial_t\hat{e}_\mathrm{v} \\ &\nabla\cdot\boldsymbol{\zeta} = -\hat{e}_\mathrm{v}\cdot[\nabla\times\hat{e}_\mathrm{v}] \\ &\nabla\times\boldsymbol{\zeta} = \hat{e}_\mathrm{v}[\nabla\cdot\hat{e}_\mathrm{v}] - [\hat{e}_\mathrm{v}\cdot\nabla]\hat{e}_\mathrm{v} \\ &[\nabla\times\boldsymbol{\zeta}]\cdot\nabla = \nabla^2 - [\hat{e}_\mathrm{v}\cdot\nabla][\hat{e}_\mathrm{v}\cdot\nabla] = [\hat{e}_\mathrm{v}\times\nabla]\cdot[\hat{e}_\mathrm{v}\times\nabla]\end{aligned} \tag{78}$$

So that the above equation is indeed equivalent to the one-dimensional wave equation:

$$\partial_t^2 a = c^2[\hat{e}_\mathrm{v}\cdot\nabla][\hat{e}_\mathrm{v}\cdot\nabla]a \tag{79}$$

If we want to obtain the conventional 3D scalar wave equation:

$$\partial_t^2 a = c^2 \nabla^2 a \tag{80}$$

Then the simplest corresponding first order equation is:

$$\partial_t \psi + c\sigma \boldsymbol{\beta} \cdot \nabla \psi = 0 \tag{81}$$

3.5. Vector Waves

> "*Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the "Old One." I, at any rate, am convinced that He is not playing at dice. Waves in three-dimensional space whose velocity is regulated by potential energy (for example, rubber bands) . . .*"
>
> – Albert Einstein, 1926 [Einstein and Born 2005]

Next we consider vector waves (polar or axial vectors). An arbitrary polarization vector can be described by a scalar amplitude and three rotation angles. Since scalar waves require five parameters, we expect vector waves to require eight parameters. As with velocity rotations, only two angles are necessary to determine the direction of polarization, but a third angle is necessary for a local description of changes in the polarization direction.

3.5.1. Rotation of Polarization

Recall that the scalar polarization is $\dot{a} = \psi^T \sigma \psi$. We now regard this as one component of a vector: $\dot{a}_3 = \psi^T \sigma_3 \psi$. The vector **a** could be polar or axial, but we will assume an axial vector (pseudovector). The three orthogonal polarization matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\bar{\imath} & 0 & 0 \\ \bar{\imath} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\imath} \\ 0 & 0 & \bar{\imath} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{82}$$

The symbol ($\bar{\imath}$) is a unit scalar imaginary which is even under spatial inversion since the spin is a pseudovector.

These matrices have the same commutation relations as the Pauli matrices and the velocity matrices (σ_i):

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \;;\; \sigma_i \sigma_j - \sigma_j \sigma_i = 2\varepsilon_{ijk} \sigma_k \tag{83}$$

The rotation operators for this space are similar to the velocity matrix rotation operators:

$$R_{\sigma_j}(\xi_i) = \exp\left(-\bar{\imath}\,\sigma_i\xi_i/2\right)\sigma_j \exp\left(\bar{\imath}\,\sigma_i\xi_i/2\right) = \sigma_j \cos\xi_i - \frac{\bar{\imath}}{2}\left[\sigma_i\sigma_j - \sigma_j\sigma_i\right]\sin\xi_i \tag{84}$$

We could simply generalize the wave function to be:

$$\psi = \dot{a}_0^{1/2} \exp\left(-\bar{\imath}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(-\tilde{\imath}\,\boldsymbol{\beta}\cdot\boldsymbol{\zeta}/2\right)\exp\left(\beta_3\sigma\alpha/2\right)\psi_0 \tag{85}$$

We might then attempt the interpretation:

$$\partial_i S_j = -\psi^\dagger \beta_i \sigma_j \psi \tag{86}$$

However, there are nine tensor components (plus three components of the time derivative) and only eight independent components of the bispinor. Therefore this interpretation is not satisfactory.

Instead, we will assume a single rotation operator for both wave velocity and polarization. Since the one-dimensional velocity is $v = c\beta_3\sigma_3$, the three-dimensional velocity for vector waves is $\mathbf{v} = c\beta_3\boldsymbol{\sigma}$. The $\boldsymbol{\beta}$ matrices which described velocity for scalar waves now represent directions relative to velocity, with β_3 representing the parallel direction. This notation is called the "chiral representation" of velocity.

Alternatively, we could associate any of the matrices β_i with velocity by rotating in the relative-velocity space of $\boldsymbol{\beta}$ matrices. Such a rotation is called a change of "representation". The form $\mathbf{v} = c\beta_1\boldsymbol{\sigma}$ has the form originally used by Dirac, and we will use these matrices for velocity. Historically, a different notation has been used for the $\boldsymbol{\beta}$ matrices. Instead of ($\beta_1, \beta_2, \beta_3$), these matrices have been called ($\gamma^5, \gamma^4, \gamma^0$). However, we will continue to call them $\boldsymbol{\beta}$ matrices except when comparing with standard results from other literature.

3.5.2. Factorization and First-Order Wave Equation

The three-dimensional bispinor wave function may have a Lorentz boost with arbitrary magnitude and direction, and may also be rotated by an arbitrary angle $\boldsymbol{\xi}$. These operators are contained in the factorization:

$$\psi = \dot{a}_0^{1/2} \exp\left(-\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(\beta_1\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2\right)\psi_0 \tag{87}$$

The wave function has seven free parameters: an amplitude, three rotation angles, and three velocity parameters. There is one additional degree of freedom which determines the definition of the relative directions β_2 and β_3. These are defined with respect to the velocity axis by the operator $\exp\left(\tilde{\imath}\,\beta_1\zeta/2\right)\psi_0$, so that the wave function is [Hestenes]:

$$\psi = \dot{a}_0^{1/2} \exp\left(-\bar{\imath}\,\boldsymbol{\sigma}\cdot\boldsymbol{\xi}/2\right)\exp\left(\beta_1\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2\right)\exp\left(\tilde{\imath}\,\beta_1\zeta/2\right)\psi_0 \tag{88}$$

Now we would like to know the equation of evolution of the wave function. Generalizing the scalar wave equation (73) to include arbitrary gradient direction yields:

$$\partial_t\psi = -c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi \tag{89}$$

These terms account for wave propagation in an arbitrary direction.

To see the wave equation in terms of observables, multiply $\psi^\dagger\sigma_j$ and add the transpose equation to obtain the time derivative of the polarization:

$$\partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = -c\nabla\left[\psi^\dagger\beta_1\psi\right] - \bar{\imath}\,c\left[\nabla\psi^\dagger\times\beta_1\boldsymbol{\sigma}\psi + \psi^\dagger\beta_1\boldsymbol{\sigma}\times\nabla\psi\right] \tag{90}$$

The terms in this equation are naturally associated with spinors by the following definitions:

$$\begin{aligned}
&\partial_t^2 a_j \equiv \partial_t\left[\psi^\dagger\sigma_j\psi\right] \\
&c^2\partial_j\left[\nabla\cdot\mathbf{a}\right] \equiv -c\partial_j\left[\psi^\dagger\beta_1\psi\right] \\
&c^2\left\{\nabla\times\nabla\times\mathbf{a}\right\}_j \equiv -\bar{\imath}\,c\varepsilon_{ijk}\left\{\partial_i\psi^\dagger\beta_1\sigma_k\psi - \psi^\dagger\beta_1\sigma_k\partial_i\psi\right\}
\end{aligned} \tag{91}$$

These identifications yield the wave equation:

$$\partial_t^2\mathbf{a} = c^2\nabla^2\mathbf{a} \tag{92}$$

The interpretation of the spinor wave functions must be self-consistent. For example, the first two identifications require:

$$\partial_t\left[\psi^\dagger\beta_1\psi\right] + c\nabla\cdot\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = 0 \tag{93}$$

This relation is easily derived from equation (89).

Also from (89):

$$\partial_t\left[\psi^\dagger\psi\right] + c^2\nabla\cdot\left[\psi^\dagger\beta_1\boldsymbol{\sigma}\psi\right] = 0 \tag{94}$$

This is the quantum mechanical continuity equation. This is the three dimensional generalization of the 1-D equation:

$$\partial_t\left|\partial_t a_F\right| + \partial_t\left|\partial_t a_B\right| + c^2\partial_z\left|\partial_z a_F\right| - c^2\partial_z\left|\partial_z a_B\right| = 0 \tag{95}$$

3.5.3. Convection and Rotation

Adding terms for convection and rotation to the bispinor wave equation yields:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi - \mathbf{u} \cdot \nabla \psi - \mathbf{w} \cdot \partial_{\boldsymbol{\varphi}} \psi \tag{96}$$

From the wave factorization we can substitute minus the angular derivative (for passive rotation) in the final term:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi - \mathbf{u} \cdot \nabla \psi - \bar{\mathrm{i}}\, \mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \psi \tag{97}$$

To see the wave equation in terms of observables, multiply $\psi^\dagger \sigma_j$ and add the transpose equation to obtain the time derivative of the polarization:

$$\partial_t \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = -c\nabla\left[\psi^\dagger \beta_1 \psi\right] + \bar{\mathrm{i}}\, c\varepsilon_{ijk}\left[\partial_i \psi^\dagger \beta_1 \sigma_k \psi - \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right] \hat{\mathbf{e}}_j - \mathbf{u} \cdot \nabla\left[\psi^\dagger \boldsymbol{\sigma} \psi\right] + \mathbf{w} \times \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] \tag{98}$$

These identifications yield the equation of a wave propagating in a moving medium:

$$\partial_t^2 \mathbf{a} = c^2 \nabla^2 \mathbf{a} - \mathbf{u} \cdot \nabla \dot{\mathbf{a}} + \mathbf{w} \times \dot{\mathbf{a}} \tag{99}$$

Using equation (97) now yields different continuity conditions:

$$\partial_t \left[\psi^\dagger \beta_1 \psi\right] + c^2 \nabla \cdot \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] + \mathbf{u} \cdot \nabla\left[\psi^\dagger \beta_1 \psi\right] = 0 \tag{100}$$

Consistency with our definition of variables requires that:

$$\mathbf{u} \cdot \nabla\left[\psi^\dagger \beta_1 \psi\right] = 0 \tag{101}$$

Also from (97):

$$\partial_t \left[\psi^\dagger \psi\right] + c^2 \nabla \cdot \left[\psi^\dagger \beta_1 \boldsymbol{\sigma} \psi\right] + \mathbf{u} \cdot \nabla\left[\psi^\dagger \psi\right] = 0 \tag{102}$$

The continuity equation now includes an additional convection term.

Next, we will interpret the wave polarization.

3.6. Waves in an Elastic Solid

"I am never content until I have constructed a mechanical model of the subject I am studying. If I succeed in making one, I understand; otherwise I do not."

– William Thomson (Lord Kelvin) 1904 [Baltimore Lectures on Molecular Dynamics and The Wave Theory of Light]

3.6.1. Basic Assumptions

We make the following basic assumptions:

1. The elastic solid is characterized by an inertial density ρ and coefficient of elasticity μ, with characteristic wave speed $c = \sqrt{\mu/\rho}$.
2. There is a linear response to variations of orientation angle $\boldsymbol{\Theta}$ relative to equilibrium. This means that an initial static perturbation (with velocity **u**=0) would yield the response:

$$\partial_t^2 \boldsymbol{\Theta} = c^2 \nabla^2 \boldsymbol{\Theta} \quad (\text{if } \mathbf{u} = 0) \tag{103}$$

3. The velocity field **u** has no compression $\nabla \cdot \mathbf{u} = 0$. Therefore the velocity may be written as the curl of a vector field:

$$\mathbf{u} = \frac{1}{2\rho}\left[\nabla \times \bar{\mathbf{J}}\right] \tag{104}$$

The vector field $\bar{\mathbf{J}}$ is called the conjugate angular momentum density. It differs from the usual definition of angular momentum density $\mathbf{J} = \mathbf{r} \times \rho\mathbf{u}$ in that it is independent of the choice of origin and can have arbitrary direction. If |**u**| falls to zero sufficiently rapidly toward infinity, then kinetic energy may be expressed as $K = \int dr^3\, \rho u^2/2 = \int dr^3\, \mathbf{w} \cdot \bar{\mathbf{J}}/2$, where $\mathbf{w} = \nabla \times \mathbf{u}/2$ is the angular velocity, or vorticity (for rigid rotations, |**u**| does not vanish at infinity and the relation between **u** and $\bar{\mathbf{J}}$ has opposite sign). Hence $\bar{\mathbf{J}}$ is the variable conjugate to angular velocity for a Lagrangian which depends on **u** only through the (positive) kinetic energy.

Additional assumptions will be introduced in order to simplify the mathematics, and these may limit the generality of the results.

3.6.2. Equation of Evolution

Starting from (103), we define an angular potential **Q** such that:

$$\nabla^2 \mathbf{Q} = -4\rho\boldsymbol{\Theta} \tag{105}$$

The static condition for **Q** is:

$$\nabla^2 \left\{\partial_t^2 \mathbf{Q} - c^2 \nabla^2 \mathbf{Q}\right\} = 0 \quad (\text{if } \mathbf{u} = 0) \tag{106}$$

Define the spin angular momentum as:

$$\mathbf{S} \equiv \partial_t \mathbf{Q} \tag{107}$$

The static condition is then:

$$\nabla^2\left\{\partial_t \mathbf{S} - c^2\nabla^2\mathbf{Q}\right\} = 0 \quad (\text{if } \mathbf{u} = 0) \tag{108}$$

When motion is present, it contributes to the time derivative only through convection ($-\mathbf{u}\cdot\nabla\mathbf{S}$) and rotation ($\mathbf{w}\times\mathbf{S}$):

$$\nabla^2\left\{\partial_t \mathbf{S} - c^2\nabla^2\mathbf{Q} + \mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{w}\times\mathbf{S}\right\} = 0 \tag{109}$$

This assumes that there are no velocity-dependent forces such as frictional damping. From here on, we will consider only wave-like solutions satisfying:

$$\partial_t \mathbf{S} - c^2\nabla^2\mathbf{Q} + \mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{w}\times\mathbf{S} = 0 \tag{110}$$

For oscillatory solutions to this equation, the first two terms are always in phase ($\partial_t^2\mathbf{Q} - c^2\nabla^2\mathbf{Q}$), whereas the velocity-dependent terms may have different phase. However, if the velocity-dependent terms do not add to zero then they must have the same phase as the linear terms:

$$\mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{w}\times\mathbf{S} = \Omega^2(\mathbf{r})\mathbf{Q} \tag{111}$$

where $\Omega^2(\mathbf{r})$ is some function of position (more generally, $\Omega^2(\mathbf{r})$ could have different values for each component of **Q**). Substitution yields:

$$\partial_t^2\mathbf{Q} - c^2\nabla^2\mathbf{Q} + \Omega^2(\mathbf{r})\mathbf{Q} = 0 \tag{112}$$

If $\Omega^2(\mathbf{r})$ is constant and positive, then this is the Klein-Gordon equation, which is ordinarily associated with bosons.

Now our only remaining task is to solve for the velocity in terms of other wave variables. To do this we note that, as discussed above, the wave equation (8) can be written in terms of a four-component complex Dirac bispinor (ψ) using the following identifications:

$$\begin{aligned}
&\partial_t^2 Q_j \equiv \frac{1}{2}\partial_t\left[\psi^\dagger\sigma_j\psi\right] \\
&c^2\partial_j\left[\nabla\cdot\mathbf{Q}\right] \equiv -\frac{1}{2}c\partial_j\left[\psi^\dagger\beta_1\psi\right] \\
&c^2\left\{\nabla\times\nabla\times\mathbf{Q}\right\}_j \equiv -\frac{\mathrm{i}}{2}c\varepsilon_{ijk}\left\{\partial_i\psi^\dagger\beta_1\sigma_k\psi - \psi^\dagger\beta_1\sigma_k\partial_i\psi\right\}
\end{aligned} \tag{113}$$

The matrices $c\beta_1\sigma_j$ are the Dirac velocity matrices, more conventionally denoted as $c\gamma^5\sigma_j$.

The above identifications provide 7 constraints on the 8 free parameters of the Dirac bispinor. In terms of bispinors, the rotational wave equation (109) is:

$$\frac{\partial}{\partial t}\left[\psi^\dagger \sigma_j \psi\right] + c\partial_j\left[\psi^\dagger \beta_1 \psi\right] - \mathrm{i}\, c\varepsilon_{ijk}\left\{\partial_i \psi^\dagger \beta_1 \sigma_k \psi + \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right\} + \mathbf{u}\cdot\nabla\left[\psi^\dagger \sigma_j \psi\right] - \varepsilon_{kij} w_k \left[\psi^\dagger \sigma_i \psi\right] = 0 \tag{114}$$

Expanding the derivatives yields:

$$\psi^\dagger \sigma_j \left[\partial_t \psi + c\beta_1 \boldsymbol{\sigma}\cdot\nabla\psi + \mathbf{u}\cdot\nabla\psi + \mathbf{w}\cdot\frac{\mathrm{i}\boldsymbol{\sigma}}{2}\psi\right] + \text{c.c.} = 0 \tag{115}$$

where (c.c.) represents the complex conjugate. The Hermitian conjugate wave function may be regarded as an independent variable (the independent real and imaginary parts of the wave function are linear combinations of elements of ψ and $\psi^\dagger$). Validity for arbitrary $\psi^\dagger$ requires the terms in brackets to sum to zero. This yields the Dirac equation:

$$\partial_t \psi + c\beta_1 \boldsymbol{\sigma}\cdot\nabla\psi + \mathbf{u}\cdot\nabla\psi + \mathbf{w}\cdot\frac{\mathrm{i}\boldsymbol{\sigma}}{2}\psi + \mathrm{i}\,\chi\psi = 0 \tag{116}$$

where χ may be any operator with the property:

$$\mathrm{Re}\left(\psi^\dagger \sigma_j\, \mathrm{i}\,\chi\psi\right) = 0 \tag{117}$$

Since χ has no effect on physical quantities, we assume it to be zero.

Now we construct a Lagrange density. Lagrange's equations of motion for a field variable ψ are:

$$\partial_t \frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]} + \sum_j \partial_j \frac{\partial\mathcal{L}}{\partial\left[\partial_j\psi\right]} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \tag{118}$$

A similar equation holds with $\psi^\dagger$ replacing ψ. The Lagrange density for rotational waves is therefore:

$$\mathcal{L} = \mathrm{Re}\left\{-\mathrm{i}\psi^\dagger\partial_t\psi + \psi^\dagger\left[-\mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla\right]\psi + \psi^\dagger\left[-\mathrm{i}\,\mathbf{u}\cdot\nabla + \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\right]\psi\right\} \tag{119}$$

Using the Hermitian conjugate of (118) yields simply $\partial\mathcal{L}/\partial\psi^\dagger = 0$. The conjugate momentum to the field $\psi^\dagger$ is p_ψ:

$$p_\psi = \frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]} = -\mathrm{i}\psi^\dagger \tag{120}$$

We assume that we can neglect boundary terms in the integration by parts of:

$$\int dr^3 \mathbf{w}\cdot\mathbf{S} \equiv \int dr^3 [\nabla\times\mathbf{u}]\cdot\mathbf{S} = \int dr^3 \mathbf{u}\cdot[\nabla\times\mathbf{S}] \tag{121}$$

The conjugate momentum for **r** is:

$$\mathbf{p_r} = \frac{\partial\mathcal{L}}{\partial\mathbf{u}} = \mathrm{Re}\left(-\psi^\dagger \mathrm{i}\nabla\psi\right) + \frac{\nabla\times\mathbf{S}}{2} = \rho\mathbf{u} \tag{122}$$

This conjugate momentum was derived under the assumption that **u** is an independent variable. Identification of $\mathbf{p_r}$ with $\rho\mathbf{u}$ is justified by the lack of external forces, implying that $\rho\mathbf{u}$ enters the Lagrangian only through the kinetic energy.

Making **u** a function of ψ introduces a factor of $1/2$:

$$\mathcal{L} = \mathrm{Re}\left\{-\mathrm{i}\psi^\dagger\partial_t\psi + \psi^\dagger\left[-\mathrm{i}c\beta_1\boldsymbol{\sigma}\cdot\nabla\right]\psi\right\} + \frac{1}{2\rho}\left[\mathrm{Re}\left(-\psi^\dagger \mathrm{i}\nabla\psi\right) + \frac{1}{2}\nabla\times\psi^\dagger\frac{\boldsymbol{\sigma}}{2}\psi\right]^2 \tag{123}$$

Variation of this Lagrange density determines the evolution of rotational waves:

$$\partial_t\psi + c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi + \mathbf{u}\cdot\nabla\psi + \mathrm{i}\,\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}\psi = 0 \tag{124}$$

where velocity **u** and vorticity $\mathbf{w} = \nabla\times\mathbf{u}/2$ are determined from (20).

3.6.3. Dynamical Variables

3.6.3.1. Angular Momentum

Recall that the velocity is the curl of an origin-independent angular momentum:

$$\mathbf{u} = \frac{1}{2\rho}\nabla\times\bar{\mathbf{J}} = \frac{1}{2\rho}\nabla\times\left[\bar{\mathbf{L}} + \mathbf{S}\right] \tag{125}$$

We now identify the orbital ($\mathbf{u}_\mathrm{L}$) and spin ($\mathbf{u}_\mathrm{S}$) contributions to velocity:

$$\begin{aligned} \frac{1}{2}\nabla\times\bar{\mathbf{L}} &= -\mathrm{Re}\left(\psi^\dagger \mathrm{i}\nabla\psi\right) = \rho\mathbf{u}_\mathrm{L} \\ \frac{1}{2}\nabla\times\mathbf{S} &= \rho\mathbf{u}_\mathrm{S} \end{aligned} \tag{126}$$

The incompressibility condition $\nabla\cdot\mathbf{u} = 0$ places an additional restriction on the wave function:

$$\nabla\cdot\rho\mathbf{u}_L = \frac{1}{2}\nabla\cdot\left[-\psi^\dagger \mathrm{i}\nabla\psi + \mathrm{i}[\nabla\psi]^\dagger\psi\right] = \frac{\mathrm{i}}{2}\left\{-\psi^\dagger\nabla^2\psi + \left[\nabla^2\psi^\dagger\right]\psi\right\} = 0 \tag{127}$$

The usual definition of angular momentum depends on the choice of origin. If we define a rotational velocity as $\mathbf{u}_R = \partial_t \boldsymbol{\varphi} \times \mathbf{r}$ and note that the vorticity is the instantaneous angular velocity ($\mathbf{w} = \partial_t \boldsymbol{\varphi}$), then from (119) the conjugate angular momentum would be:

$$\mathbf{p}_{\boldsymbol{\varphi}} = \frac{\partial \mathcal{L}}{\partial[\partial_t \boldsymbol{\varphi}]} = \psi^\dagger \left\{ -\mathbf{r} \times \mathrm{i}\nabla + \frac{1}{2}\boldsymbol{\sigma} \right\} \psi = \mathbf{L} + \mathbf{S} \tag{128}$$

The quantity $\mathbf{L} = \psi^\dagger \{-\mathbf{r} \times \mathrm{i}\nabla\} \psi$ is called the 'orbital' angular momentum, while $\mathbf{S} = \frac{1}{2}\psi^\dagger \boldsymbol{\sigma} \psi$ is called the 'spin' angular momentum. Historically, the existence of 'spin' angular momentum has defied explanation, yet we have derived it from a simple classical model.

3.6.3.2. Energy and momentum

In the Lagrange density defined above, the velocity-dependent term is clearly the kinetic energy density. This observation suggests that the Lagrangian has the form:

$$\mathcal{L} = \mathrm{Re}\left\{ -\mathrm{i}\psi^\dagger \partial_t \psi + \psi^\dagger [-\mathrm{i}c\beta_1 \boldsymbol{\sigma} \cdot \nabla] \psi + \frac{1}{2}\rho u^2 \right\} = -\mathcal{E} + U + K \tag{129}$$

where $\mathcal{E} = \mathrm{Re}\left(\mathrm{i}\psi^\dagger \partial_t \psi\right)$ is the total energy, $U = \mathrm{Re}\left(-\psi^\dagger [\mathrm{i}c\beta_1 \boldsymbol{\sigma} \cdot \nabla] \psi\right)$ is potential energy, and K is kinetic energy density.

The Hamiltonian is the negative of the energy:

$$\mathcal{H} = p_\psi \partial_t \psi - \mathcal{L} = \psi^\dagger \left\{ \mathrm{i}c\beta_1 \boldsymbol{\sigma} \cdot \nabla + \mathrm{i}\mathbf{u} \cdot \nabla - \mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \right\} \psi = -\left\{ U + \frac{1}{2}\rho u^2 \right\} \tag{130}$$

Hamilton's equation for the wave function is:

$$\partial_t \psi = \frac{\partial \mathcal{H}}{\partial p_\psi} = \frac{\partial \mathcal{H}}{\partial\left[-\mathrm{i}\psi^\dagger\right]} = \left\{ -c\beta_1 \boldsymbol{\sigma} \cdot \nabla - \mathbf{u} \cdot \nabla - \mathrm{i}\mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \right\} \psi \tag{131}$$

We can also define a Hamiltonian operator with $\partial_t \psi = \mathrm{i} H \psi$ (note opposite sign convention from quantum mechanics):

$$H = \mathrm{i}c\gamma^5 \boldsymbol{\sigma} \cdot \nabla + \mathrm{i}\mathbf{u} \cdot \nabla - \mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \tag{132}$$

The Hamiltonian is a special case (T_0^0) of the energy-momentum tensor:

$$T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \psi\right]} \partial_\nu \psi - \mathcal{L}\delta_\nu^\mu \tag{133}$$

The dynamical momentum density is derived from the orbital part of the angular momentum:

$$T_i^0 = P_i = -\mathrm{i}\,\psi^\dagger \partial_i \psi \tag{134}$$

This is identical to the momentum of relativistic quantum mechanics.

The sign of the Hamiltonian (and Lagrangian) is simply a convention. In analogy with plane waves, the function $\cos(\omega t - kx)$ with $\omega > 0$ is equivalent to $\cos(\omega t + kx)$ with $\omega < 0$. We could change the sign of the Lagrangian and Hamiltonian and still preserve the sign of the momentum by using covariant derivatives ($\partial_\mu = (\partial_t, -\partial_i)$). However, such a relativistic construction is unnecessary and perhaps misleading since we are in fact dealing with Galilean space-time. The energy-momentum tensor components above $T_\mu^0 = (-\mathcal{E}, P_i)$ may still be regarded as a covariant vector whose magnitude is the scalar $\mathcal{E}^2 - P^2$. The equation of evolution is of course unaffected by the choice of sign of the Lagrangian.

3.7. Electron Waves

> *"... a great step would be made when we should be able to say of electricity that which we say of light, in saying that it consists of undulations."*
>
> — Sir George Gabriel Stokes, 1879

3.7.1. Free Electron Equation

The bispinor equation for angular momentum density is:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi - \mathbf{u} \cdot \nabla - \mathrm{i}\,\mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \psi \tag{135}$$

A formal solution is:

$$\psi(\mathbf{r}, t) = \exp\left\{ \int_{t_0}^{t} dt \left(-c\beta_1 \boldsymbol{\sigma} \cdot \nabla - \mathbf{u} \cdot \nabla - \mathrm{i}\,\mathbf{w} \cdot \frac{\boldsymbol{\sigma}}{2} \right) \right\} \psi(\mathbf{r}, t_0) \tag{136}$$

3.7.1.1. Mass, Convection, and Rotation

Dirac's derivation of the mass term simply required that each component of the wave function satisfy the Klein-Gordon equation. One possible formulation would be:

$$\partial_t \psi = -c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi - \mathrm{i}\,\mu \beta_1 \boldsymbol{\sigma} \cdot \left[\hat{\mathbf{e}}_3 \times \nabla\right] \psi \tag{137}$$

The second-order equation is:

$$\partial_t^2 \psi = \left[c^2 \nabla^2 \psi - \mu^2 \left[\hat{\mathbf{e}}_3 \times \nabla\right]^2\right] \psi \tag{138}$$

which is equivalent to Klein-Gordon if the wave function is an eigenfunction of the operator $\left[\hat{\mathbf{e}}_3 \times \nabla\right]^2$. The equivalent classical equation is:

$$\begin{aligned} &\partial_t \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = -c\nabla\left[\psi^\dagger \beta_1 \psi\right] + \bar{\mathrm{i}}\, c\varepsilon_{ijk}\left[\partial_i \psi^\dagger \beta_1 \sigma_k \psi - \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right] \hat{\mathbf{e}}_j \\ &+ \mu\left[\hat{\mathbf{e}}_3 \times \nabla\right] \times \psi^\dagger \beta_1 \boldsymbol{\sigma} \psi \end{aligned} \tag{139}$$

which can be equivalent to Klein-Gordon if $\mu\left[\hat{\mathbf{e}}_3 \times \nabla\right] \times \psi^\dagger \beta_1 \boldsymbol{\sigma} \psi = -\Omega^2 \psi^\dagger \boldsymbol{\sigma} \psi$.

3.7.1.2. Dirac Equation

Dirac's choice of mass term differs from the one above:

$$\partial_t \psi + c\beta_1 \boldsymbol{\sigma} \cdot \nabla \psi = -\mathrm{i}\,\Omega \beta_3 \psi \tag{140}$$

where $\Omega = m_e c^2 / \hbar$. Other representations of this equation are:

$$\begin{aligned} &\partial_t \psi + c\boldsymbol{\alpha} \cdot \nabla \psi = -\mathrm{i}\,\Omega \beta \psi \quad \left(\text{Dirac's original notation}\right) \\ &\gamma^0 \partial_t \psi + c\gamma^0 \gamma^5 \boldsymbol{\sigma} \cdot \nabla \psi \equiv \gamma^\mu \partial_\mu \psi = -\mathrm{i}\,\Omega \psi \quad \left(\text{Relativist ic quantum mechanics notation}\right) \end{aligned} \tag{141}$$

In quantum mechanics, Planck's constant $\hbar$ appears explicitly in the operators and the wave function is normalized to one for the purpose of computing correlations. However, physically it is more sensible to normalize the wave function to $\hbar$ so that it is clear that the wave function describes the evolution of angular momentum density. One can still compute correlations, of course, as we will see later. For consistency with traditional quantum mechanics, we will include the factor of $\hbar$ in our equations.

The equation for spin angular momentum density is simply:

$$\partial_t \left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = -c\nabla\left[\psi^\dagger \beta_1 \psi\right] + \bar{\mathrm{i}}\, c\varepsilon_{ijk}\left[\partial_i \psi^\dagger \beta_1 \sigma_k \psi - \psi^\dagger \beta_1 \sigma_k \partial_i \psi\right] \tag{142}$$

which we interpret as an ordinary wave equation (the convection and rotation terms are presumed to cancel):

$$\partial_t^2 \mathbf{Q} = c^2 \nabla\left[\nabla \cdot \mathbf{Q}\right] - c^2 \nabla \times \left[\nabla \times \mathbf{Q}\right] = c^2 \nabla^2 \mathbf{Q} \tag{143}$$

Dirac's choice of mass term eliminates the mass from the second-order wave equation. One consequence of this choice is that the rationale for quantization via soliton waves is lost. So while Dirac's equation can be used in describing particle motion and interactions, it cannot explain the existence of discrete particles.

Dirac also assumed that stationary states have the form:

$$\partial_t \psi = -\mathrm{i}\frac{E}{\hbar}\psi \tag{144}$$

which has the formal solution:

$$\psi(\mathbf{r},t) = \exp\left\{-\mathrm{i}\frac{E}{\hbar}(t-t_0)\right\}\psi(\mathbf{r},t_0) \tag{145}$$

This solution is puzzling because the phase variation represented by the energy eigenvalue *E* does not correspond to any actual oscillation in real space. The phase simply cancels out when computing observables. A more reasonable starting point would be to neglect gradients in () to get:

$$\psi(\mathbf{r},t) = \exp\left\{-\mathrm{i}\,\mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2}(t-t_0)\right\}\psi(\mathbf{r},t_0) \tag{146}$$

If the wave function is a spin eigenfunction $(\sigma_s/2)\psi = s\psi$, with eigenvalue *s*, then the exponent can be treated as a scalar, as in quantum mechanics. The energy eigenvalue would then represent twice the rotational energy ($E = \mathbf{w}\cdot\mathbf{S}$), consistent with an equipartition of energy between kinetic and potential energy. In this case there would also be no real oscillation. However, we can make this result sensible by assuming it to be an approximation. We suppose that the wave function is not exactly an eigenfunction of spin, so that there are oscillations in real space. For example, the spin direction may rotate at a rate small compared to the magnitude of angular velocity. For example, one can envision concentric spherical shells wobbling rigidly so that the top and bottom points from the equilibrium position rotate in circles about the z-axis, yielding a net average angular momentum. But we assume that the approximation of spin eigenfunctions is valid for the purposes of computing eigenvalues and correlations between states.

Considering the lack of real oscillation in conventional quantum mechanics, it is interesting to note that physicists in the nineteenth century, led by William Thomson (Lord Kelvin), proposed a model of vacuum as consisting of a fluid filled with vortices. This model is called the vortex sponge, and still has its adherents today. The model is also has relevance to the behavior of liquid helium. This model would eliminate the requirement of oscillation, since steady flows are possible in a fluid. The model can also produce shear waves propagating among the vortices. But the model is conceptually more complex that the elastic solid, so we will not pursue it here.

If we neglect gradients in the electron equation, we have:

$$E\psi = \hbar\Omega\beta_3\psi \tag{147}$$

which has solutions: $\psi=[1\ 0\ 0\ 0]^T$ and $\psi=[0\ 1\ 0\ 0]^T$ for $E=\hbar\,\Omega$, and $\psi=[0\ 0\ 1\ 0]^T$ and $\psi=[0\ 0\ 0\ 1]^T$ for $E=-\hbar\,\Omega$. For each sign of E, the two solutions differ in the sign of the x_3-component of spin. These solutions are referred to as "spin-up" and "spin-down" solutions. The positive and negative signs of E are assumed to correspond to matter and anti-matter, respectively. We will now examine the relationship between matter and anti-matter further.

3.7.1. Angular separation

Recall Dirac's equation for a free particle:

$$\partial_t\psi = -c\beta_1\boldsymbol{\sigma}\cdot\nabla\psi - \mathrm{i}\,\Omega\beta_3\psi \tag{148}$$

The operator $\boldsymbol{\sigma}\cdot\nabla\psi$ can be factored:

$$\boldsymbol{\sigma}\cdot\nabla\psi = \sigma_r\left[\partial_r + \mathrm{i}\frac{\boldsymbol{\sigma}}{r}\cdot\left[\mathbf{r}\times\nabla\right]\right]\psi = \sigma_r\left[\partial_r - \frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right]\psi \tag{149}$$

The two-component angular solutions of the eigenvalue equations $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(+)}_{l,m} = l = -1+\kappa$ and $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(-)}_{l,m} = -\left[l+2\right] = -1-\kappa$ are well known (*e.g.* (Bjorken and Drell 1964)), and are derived in the Appendix. These two angular solutions are related by $\sigma_r\Phi^{(+)}_{l,m} = \Phi^{(-)}_{l,m}$ and yield opposite eigenvalues of the parity (spatial inversion) operation.

These angular solutions may be combined to form two independent wave functions:

$$\psi^{(+)} = \frac{1}{r}\begin{bmatrix}\tilde{\mathrm{i}}\,G\Phi^{(+)}_{l,m}\\ F\Phi^{(-)}_{l,m}\end{bmatrix} \quad \text{or} \quad \psi^{(-)} = \frac{1}{r}\begin{bmatrix}\tilde{\mathrm{i}}\,F\Phi^{(-)}_{l,m}\\ G\Phi^{(+)}_{l,m}\end{bmatrix} \tag{150}$$

3.7.2. Velocity Rotation and Mass

It is instructive to compute the effect of mass on the wave velocity:

$$\begin{aligned}\frac{d}{dt}\left[\psi^{(+)\dagger}\beta_1\sigma_i\psi^{(+)}\right] &= \psi^{(+)\dagger}\beta_1\sigma_i\left[-\mathrm{i}\,\Omega\beta_3\psi^{(+)}\right] + \left[-\mathrm{i}\,\Omega\beta_3\psi^{(+)}\right]^\dagger\beta_1\sigma_i\psi^{(+)}\\ &= -2\frac{\Omega}{r^2}FG\left\{\Phi^{(+)\dagger}_{l,m}\sigma_i\Phi^{(-)}_{l,m} + \Phi^{(-)\dagger}_{l,m}\sigma_i\Phi^{(+)}_{l,m}\right\}\\ &= 2\Omega\frac{FG}{r^2}\left\{\Phi^{(+)\dagger}_{l,m}\left[\sigma_i\sigma_r + \sigma_r\sigma_i\right]\Phi^{(+)}_{l,m}\right\}\\ &= 4\Omega\frac{FG}{r^2}\delta_{ir}\left\{\Phi^{(+)\dagger}_{l,m}\Phi^{(+)}_{l,m}\right\}\end{aligned} \tag{151}$$

The mass term represents a radial acceleration of the wave, which is inward provided that the appropriate sign is chosen for Ω. This result implies circular propagation, consistent with the explanation of the relativistic mass-energy relation given in Chapter 1.

3.7.3. Wave Interference and Potentials

Next we investigate the origin of electromagnetic potentials. Certain observables (scalars and vectors) should be additive when two waves are superposed. This implies that when two waves ψ_A and ψ_B are superposed, the total wave ψ_T has the property that:

$$\psi_T^\dagger G \psi_T = \psi_A^\dagger G \psi_A + \psi_B^\dagger G \psi_B \tag{152}$$

for some linear Hermitian operator *G*. If we simply added the two wave functions, we would have instead:

$$\left[\psi_A^\dagger + \psi_B^\dagger\right] G \left[\psi_A + \psi_B\right] = \psi_A^\dagger G \psi_A + \psi_B^\dagger G \psi_B + \psi_A^\dagger G \psi_B + \psi_B^\dagger G \psi_A \tag{153}$$

The additional terms are clearly not zero in general. However, they can be forced to zero by introducing phase shifts to the wave functions. Using a subscript zero to represent each wave function in the absence of interference, let:

$$\begin{aligned} \psi_A &= \exp\left(-\mathrm{i}\left[\delta_B + \pi\right]/2\right)\psi_{A0} \\ \psi_B &= \exp\left(-\mathrm{i}\,\delta_A/2\right)\psi_{B0} \end{aligned} \tag{154}$$

The relative phase shift π could be distributed between the two waves or incorporated into δ_A and δ_B, but we will treat ψ_A as the 'test wave' and ψ_B as the 'source wave' and require the condition below to hold even with δ_A and δ_B equal to zero. Linear addition of the observable *G* requires:

$$\psi_A^\dagger G \psi_B + \psi_B^\dagger G \psi_A = 0 \tag{155}$$

If either wave function is an eigenfunction of some additive observable such as spin ($G\psi_A = \lambda\psi_A$ or $G\psi_B = \lambda\psi_B$ for some scalar λ), then this result reduces to:

$$\psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A = 0 \tag{156}$$

In terms of the unperturbed wave functions:

$$\psi_{A0}^\dagger \exp\left(-\mathrm{i}\left[\pi + \delta_A - \delta_B\right]/2\right)\psi_{B0} + \psi_{B0}^\dagger \exp\left(\mathrm{i}\left[\pi + \delta_A - \delta_B\right]/2\right)\psi_{A0} = 0 \tag{157}$$

If we interpret the quantity $\psi_A^\dagger \psi_B$ as a two-particle state, then interchanging the two particles yields:

$$\left[\psi_A^\dagger \psi_B\right]_{A\leftrightarrow B} = \psi_B^\dagger \psi_A = -\psi_A^\dagger \psi_B \tag{158}$$

This means that the two-particle state is anti-symmetric with respect to exchange of particles. This symmetry is called the Pauli Exclusion Principle because it prohibits two identical fermions from being in the same state ($\psi_A = \psi_B$ yields $\psi_A^\dagger \psi_A = -\psi_A^\dagger \psi_A$). Thus the Pauli Exclusion Principle results from the arbitrary separation of the complete wave function into two independent parts. In quantum mechanics, the two-fermion state is typically constructed as:

$$\psi_{A,B} = \frac{\psi_B^\dagger \psi_A - \psi_A^\dagger \psi_B}{\sqrt{2}} \tag{159}$$

so that the Exclusion Principle is automatically satisfied.

The constant phase shift $\pi/2$ has no effect on dynamics. However, some observables computed from these independent wave functions may differ from those of the free particle wave. For example:

$$\psi_{A0}^\dagger G \psi_{A0} = \psi_A^\dagger \left[\exp(-\mathrm{i}\,\delta_B/2) G \exp(\mathrm{i}\,\delta_B/2)\right]\psi_A = \psi_A^\dagger G'_A \psi_A \tag{160}$$

Hence the effect of wave interference is to change the operator for wave packet ψ_A from G to G'_A:

$$G'_A = \exp(-\mathrm{i}\,\delta_B/2) G \exp(\mathrm{i}\,\delta_B/2) \tag{161}$$

Applying this rule to the operators ∂_t and H yields:

$$\left\{\partial_t + \left[\exp(-\mathrm{i}\,\delta_B/2)\partial_t \exp(\mathrm{i}\,\delta_B/2)\right]\right\}\psi_A = \exp(-\mathrm{i}\,\delta_B/2)\,\mathrm{i}\,H \exp(\mathrm{i}\,\delta_B/2)\,\psi_A \tag{162}$$

$$\frac{H}{\hbar} = \mathrm{i}\,c\beta_1 \boldsymbol{\sigma}\cdot\nabla + \mathrm{i}\,\mathbf{u}\cdot\nabla - \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2} \tag{163}$$

Substituting the general form of the Hamiltonian:

$$\begin{aligned}&\left[\partial_t + \frac{\mathrm{i}}{2}\partial_t\delta_B\right]\psi'_B + \left[c\beta_1\boldsymbol{\sigma}\cdot\nabla + \frac{\mathrm{i}}{2}c\beta_1\boldsymbol{\sigma}\cdot\nabla\delta_B\right]\psi'_B \\ &+\left[\left(\mathbf{u}_A+\mathbf{u}_B\right)\cdot\nabla + \frac{\mathrm{i}}{2}\left(\mathbf{u}_A+\mathbf{u}_B\right)\cdot\nabla\delta_B\right]\psi'_B + \mathrm{i}\left(\dot{\boldsymbol{\varphi}}_A+\dot{\boldsymbol{\varphi}}_B\right)\cdot\frac{\boldsymbol{\sigma}}{2}\psi'_B = 0\end{aligned} \tag{164}$$

Substituting the mass term for the free electron:

$$\left[\partial_t + \frac{\mathrm{i}}{2}\partial_t\delta_B\right]\psi_A + \left[c\beta_1\boldsymbol{\sigma}\cdot\nabla + \frac{\mathrm{i}}{2}c\beta_1\boldsymbol{\sigma}\cdot\nabla\delta_B\right]\psi_A$$
$$+\left[\mathbf{u}_B\cdot\nabla + \frac{\mathrm{i}}{2}(\mathbf{u}_A+\mathbf{u}_B)\cdot\nabla\delta_B\right]\psi_A + \frac{\mathrm{i}}{2}[\boldsymbol{\sigma}\cdot\dot{\boldsymbol{\phi}}_B]\psi_A + \mathrm{i}\,\Omega\beta_3\psi_A = 0 \tag{165}$$

Since we are interested in the effects of the phase shift, we will neglect the extra terms which are independent of δ_B (without explicit justification). We then define the electromagnetic potentials as:

$$\frac{e}{\hbar}\mathbf{A} \equiv -\frac{c}{2}\nabla\delta_B$$
$$\frac{e}{\hbar}\Phi \equiv \frac{1}{2}\partial_t\delta_B - \mathbf{u}\cdot\frac{e}{\hbar c}\mathbf{A} \tag{166}$$

Although the vector potential $\mathbf{A}$ is a gradient, its curl (the magnetic field) may be nonzero because δ_B is a phase angle which may be multi-valued. For example, the multi-valued function $\delta_B = \arctan(x_2/x_1)$ has gradient components:

$$\partial_1\delta_B = -\frac{x_2}{\left[x_1^2 + x_2^2\right]^{(1/2)}}$$
$$\partial_2\delta_B = \frac{x_1}{\left[x_1^2 + x_2^2\right]^{(1/2)}} \tag{167}$$

The curl of this gradient is clearly non-zero. See Kleinert (2007) for a discussion of multi-valued potentials in electromagnetism.

With these definitions, the electron equation in the presence of another wave becomes:

$$\left[\partial_t + \mathrm{i}\frac{e}{\hbar}\Phi\right]\psi_A + \left[c\beta_1\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\frac{e}{\hbar}\beta_1\boldsymbol{\sigma}\cdot\mathbf{A}\right]\psi_A + \mathrm{i}\,\Omega\beta_3\psi_A = 0 \tag{168}$$

Hence electromagnetic potentials result from wave interference under the assumption that different wave packets are independent. The above analysis is not very precise, however, as we neglected changes in medium velocity and vorticity, and did not specify which observables should be additive (total momentum density and total angular momentum density should both have this property). A complete analysis of particle interactions would require knowledge of the soliton wave functions of each particle.

Setting $\hbar\partial_t\psi_A = \mathrm{i}\,H\psi$, the modified Hamiltonian is:

$$H = -e\Phi + \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla + c\beta_1\boldsymbol{\sigma}\cdot\frac{e}{c}\mathbf{A} - \Omega\beta_3 \tag{169}$$

Multiple source waves may be treated sequentially, at least as a first approximation. For a given test wave, make it independent of the first source wave as above. Then take the modified test wave and make it independent of the second source wave. Repetition of this process for all source waves results in the addition of phase shifts or equivalently, the addition of potentials. Matter and anti-matter solutions are assumed to yield opposite signs of phase shift. One may also infer that soliton waves with identical long-range (electromagnetic) potentials (e.g. positrons and protons) also have identical bispinor wave functions at large distances from their centers.

In quantum mechanics, it is necessary to treat various wave packets as independent 'particles'. However, with a classical wave theory of matter it may be simpler to solve the single equation for the total angular momentum density, then decompose the solution into soliton 'particles' for comparison with experiment.

3.7.4. Lorenz Force

In terms of electromagnetic potentials, the modified Hamiltonian is:

$$\begin{aligned}\frac{H}{\hbar} &= \bar{q}\Phi - \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla - \beta_1\boldsymbol{\sigma}\cdot\bar{q}\mathbf{A} - \mathrm{i}\,\mathbf{u}\cdot\nabla + \mathbf{w}\cdot\frac{\boldsymbol{\sigma}}{2} \\ &= \bar{q}\Phi - \mathrm{i}\,c\beta_1\boldsymbol{\sigma}\cdot\nabla - \beta_1\boldsymbol{\sigma}\cdot\bar{q}\mathbf{A} + \Omega\beta_3\end{aligned} \tag{170}$$

Recalling the *u*-dependence of $q\Phi$ and **w** (and our change of sign of H), the conjugate momentum for **r** is now:

$$\mathbf{p}_{\mathbf{r}} = \frac{\delta L}{\delta[\mathbf{u}]} = \frac{\delta \mathrm{H}}{\delta[\mathbf{u}]} = -\psi^{\dagger}\left\{\mathrm{i}\nabla + \frac{q}{c}\mathbf{A}\right\}\psi + \rho\mathbf{u} = \mathbf{p}_0 - \frac{q}{c}\mathbf{A} + \mathbf{q} \tag{171}$$

where $\mathbf{p}_0 = \psi^{\dagger}\{-\mathrm{i}\nabla\}\psi$ is the free particle wave momentum.

The time derivative of any observable *Q* is:

$$\partial_t\left[\psi^{\dagger}Q\psi\right] = \left[\partial_t\psi^{\dagger}\right]Q\psi + \psi^{\dagger}Q\partial_t\psi + \psi^{\dagger}\left[\partial_t Q\right]\psi = \psi^{\dagger}\,\mathrm{i}\left[H,Q\right]\psi + \psi^{\dagger}\partial_t Q\psi \tag{172}$$

An example of this is the force density. Substituting the linear wave momentum for *Q* yields the Lorenz force law:

$$\begin{aligned}\partial_t\mathbf{p} &= \psi^{\dagger}\left\{\nabla\left[c\beta_1\boldsymbol{\sigma}\cdot\frac{q}{c}\mathbf{A}\right] - \nabla q\Phi - c\beta_1\boldsymbol{\sigma}\cdot\nabla\left[\frac{q}{c}\mathbf{A}\right] - \frac{q}{c}\frac{\partial}{\partial t}\mathbf{A}\right\}\psi \\ &= \psi^{\dagger}\left\{c\beta_1\boldsymbol{\sigma}\times\left[\nabla\times\frac{q}{c}\mathbf{A}\right] - \nabla q\Phi - \frac{q}{c}\frac{\partial}{\partial t}\mathbf{A}\right\}\psi = \psi^{\dagger}\left\{c\beta_1\boldsymbol{\sigma}\times\frac{q}{c}\mathbf{B} + q\mathbf{E}\right\}\psi\end{aligned} \tag{173}$$

where **E** and **B** are the usual electric and magnetic fields, respectively. Hence the Lorenz force has a straightforward interpretation in terms of classical wave interference.

3.7.5. Magnetic Moment

The equation of evolution in electromagnetic fields is:

$$\left[\partial_t + \mathrm{i}\,\bar{q}\Phi + c\beta_1\boldsymbol{\sigma}\cdot\nabla + \mathrm{i}\,\bar{q}\beta_1\boldsymbol{\sigma}\cdot\mathbf{A}\right]\psi = -\mathrm{i}\,\Omega\beta_3\psi \tag{174}$$

Using two-component spinors with $\psi = [\psi_1, \psi_2]^T$, this equation can be separated into two coupled equations:

$$\begin{aligned}\left[\partial_t + \mathrm{i}\,\bar{q}\Phi\right]\psi_1 + \left[c\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\bar{q}\boldsymbol{\sigma}\cdot\mathbf{A}\right]\psi_2 &= -\mathrm{i}\,\Omega\psi_1 \\ \left[\partial_t + \mathrm{i}\,\bar{q}\Phi\right]\psi_2 + \left[c\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\bar{q}\boldsymbol{\sigma}\cdot\mathbf{A}\right]\psi_1 &= \mathrm{i}\,\Omega\psi_2\end{aligned} \tag{175}$$

Let $\psi_1 \equiv \exp(-i\Omega t)\chi_1$ and $\psi_2 = \exp(-i\Omega t)\chi_2$. Substitution yields:

$$\begin{aligned}\left[\partial_t + \mathrm{i}\,\bar{q}\Phi\right]\chi_1 + \left[c\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\bar{q}\boldsymbol{\sigma}\cdot\mathbf{A}\right]\chi_2 &= 0 \\ \left[\partial_t + \mathrm{i}\,\bar{q}\Phi - 2\,\mathrm{i}\,M\right]\chi_2 + \left[c\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\bar{q}\boldsymbol{\sigma}\cdot\mathbf{A}\right]\chi_1 &= 0\end{aligned} \tag{176}$$

Next, assume that $\left|\left[\partial_t + \mathrm{i}\,\bar{q}\Phi\right]\chi_2\right| << \left|2\,\mathrm{i}\,\Omega\chi_2\right|$. This yields:

$$\left[\partial_t + \mathrm{i}\,\bar{q}\Phi\right]\chi_1 + \frac{\left[c\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\bar{q}\boldsymbol{\sigma}\cdot\mathbf{A}\right]\left[c\boldsymbol{\sigma}\cdot\nabla - \mathrm{i}\,\bar{q}\boldsymbol{\sigma}\cdot\mathbf{A}\right]}{2\,\mathrm{i}\,\Omega}\chi_1 = 0 \tag{177}$$

This is the Pauli equation [], which was the first equation to incorporate electron spin.

$$\mathrm{i}\,\partial_t\chi_1 = \left\{\frac{\boldsymbol{\sigma}\cdot\left[\mathrm{i}\,c\nabla - \bar{q}\mathbf{A}\right]\boldsymbol{\sigma}\cdot\left[\mathrm{i}\,c\nabla - \bar{q}\mathbf{A}\right]}{2\Omega} + \bar{q}\Phi\right\}\chi_1 \tag{178}$$

Using the commutation relations for the Pauli spin matrices:

$$\begin{aligned}&\boldsymbol{\sigma}\cdot\left[-\mathrm{i}\,c\nabla - \bar{q}\mathbf{A}\right]\boldsymbol{\sigma}\cdot\left[\mathrm{i}\,c\nabla - \bar{q}\mathbf{A}\right]\chi_1 = \sigma_i\left[-\mathrm{i}\,c\partial_i - \bar{q}A_i\right]\sigma_j\left[-\mathrm{i}\,c\partial_j - \bar{q}A_j\right]\chi_1 \\ &= \left\{\left[-\mathrm{i}\,c\partial_i - \bar{q}A_i\right]^2 + \varepsilon_{ijk}\,\mathrm{i}\,\sigma_k\left[-\mathrm{i}\,c\partial_i - \bar{q}A_i\right]\left[-\mathrm{i}\,c\partial_j - \bar{q}A_j\right]\right\}\chi_1 \\ &= \left\{\left[-\mathrm{i}\,c\partial_i - \bar{q}A_i\right]^2 - c\bar{q}\boldsymbol{\sigma}\cdot\mathbf{B}\right\}\chi_1\end{aligned} \tag{179}$$

Substitution yields:

$$\mathrm{i}\,\hbar\partial_t\chi_1 = \hbar\left\{\frac{\left[-\mathrm{i}\,c\partial_i - \bar{q}A_i\right]^2 - c\bar{q}\boldsymbol{\sigma}\cdot\mathbf{B}}{2\Omega} + \bar{q}\Phi\right\}\chi_1 \tag{180}$$

This equation is of course simply an approximate equation for two components of the Dirac wave function. Nonetheless, it is of historical importance because it was used by Pauli to include effects of electron spin. Without the spin term, the resultant scalar equation is the one Schrödinger first used to compute the hydrogen energy levels:

$$\mathrm{i}\hbar\partial_t\psi = \hbar\left\{\frac{\left[-\mathrm{i}c\partial_i - \bar{q}A_i\right]^2}{2\Omega} + (q/\hbar)\Phi\right\}\psi = \left\{\frac{p^2}{2mc^2} + q\Phi\right\}\psi \tag{181}$$

Schrödinger's equation is currently the conventional starting point in the study of quantum mechanics. Although simpler than the Dirac equation, it is far less intuitive. Both Lorentz invariance and the connection with spin angular momentum have been lost.

In a weak, uniform magnetic field with $\mathbf{A} = \mathbf{B}_0 \times \mathbf{r}/2$, we can neglect A^2 to obtain:

$$\begin{aligned}\mathrm{i}\hbar\partial_t\chi_1 &= \left\{\hbar\frac{-c^2\nabla^2 + \mathrm{i}c\bar{q}\left[\mathbf{B}_0 \times \mathbf{r}\right]\cdot\nabla - c\bar{q}\boldsymbol{\sigma}\cdot\mathbf{B}_0}{2\Omega} + q\Phi\right\}\chi_1 \\ &= \left\{\hbar\frac{-c^2\nabla^2 - c\bar{q}\mathbf{B}_0\cdot\left(\left[\mathbf{r}\times -\mathrm{i}\nabla\right] + \boldsymbol{\sigma}\right)}{2\Omega} + q\Phi\right\}\chi_1 = \left\{-\frac{\hbar^2}{2m}\nabla^2 - \frac{\hbar q}{2mc}\mathbf{B}_0\cdot\left[\mathbf{L} + 2\mathbf{s}\right] + q\Phi\right\}\chi_1\end{aligned} \tag{182}$$

The final form with the spin angular momentum operator ($\mathbf{s} = \boldsymbol{\sigma}/2$) is obtained by comparison with the angular momentum operator (128). This result is significant because it shows that, in this approximation, the coefficient of spin angular momentum is twice the coefficient of orbital angular momentum in the electron magnetic moment:

$$\boldsymbol{\mu} = -\frac{cq}{2\Omega}\left[\mathbf{L} + 2\mathbf{S}\right] \tag{183}$$

A free electron with $q = -e$, $\mathbf{L} = 0$, and $|\mathbf{S}| = 1/2$, has magnetic moment equal (within 0.1%) to the Bohr magneton $e\hbar/2mc = 5.78\times10^{-5}$ eV/T.

3.7.6. Spin Waves

Consider the equation for the evolution of spin (110):

$$\partial_t\mathbf{S} - c^2\nabla^2\mathbf{Q} + \mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{w}\times\mathbf{S} = 0 \tag{184}$$

If we neglect the spatial gradients, we have:

$$\partial_t\mathbf{S} = \mathbf{w}\times\mathbf{S} \tag{185}$$

The vorticity is given by:

$$\mathbf{w} = \frac{1}{2}\nabla\times\mathbf{u} = \frac{1}{2\rho}\nabla\times\left[-\mathrm{Re}\left(\psi^\dagger\,\mathrm{i}\nabla\psi\right) + \frac{1}{2}\nabla\times\mathbf{S}\right] \tag{186}$$

Keeping only the term involving spin yields:

$$\partial_t \mathbf{S} = \frac{1}{4\rho} \nabla \times [\nabla \times \mathbf{S}] \times \mathbf{S} = -\frac{1}{4\rho} \nabla^2 \mathbf{S} \times \mathbf{S} \tag{187}$$

This equation describes the simplest form of a 'spin wave', which is commonly observed in ferromagnetic materials.

3.7.7. Measurement Correlations

It is widely believed that the correlations between polarization measurements of entangled particles cannot be predicted classically. This belief is based on correlation predictions using an equation of the form:

$$P(\mathbf{a},\mathbf{b}) = \int A(\mathbf{a},\lambda_1,...,\lambda_n) B(\mathbf{b},\lambda_1,...,\lambda_n) \rho(\lambda_1,...,\lambda_n) d\lambda_1 .. d\lambda_n \tag{188}$$

where λ_i represent variables which describe the state of the system, $\rho(\lambda_1,...,\lambda_n)$ is the probability distribution of these variables, **a** and **b** are the measured polarization directions for the two entangled particles, *A* and *B* are the theoretical outcomes of the measurement (±1), and *P*(**a**,**b**) is the correlation.

John Bell [1964] proved that quantum correlations cannot be represented in this form. In particular, he proved that for three different measurements $A(\mathbf{a},\lambda_1,...,\lambda_n)$, $B(\mathbf{b},\lambda_1,...,\lambda_n)$, and $C(\mathbf{c},\lambda_1,...,\lambda_n)$:

$$1 + P(\mathbf{b},\mathbf{c}) \geq |P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \tag{189}$$

This condition is violated by quantum mechanical (and physically observed) correlations, which can be measured using two or more particles whose spins are constrained. For example, if a pair of spin ½ particles is produced with opposite spin, the correlation between their spin measurements by detectors oriented with relative angle φ is:

$$C_{\text{pair}}(\varphi) = -\cos\varphi \tag{190}$$

This correlation violates Bell's condition. For example, if the detectors **a**, **b**, and **c** are oriented at angles 0, $\pi/4$, and $3\pi/4$, respectively, then:

$$\begin{aligned} &C(\mathbf{a},\mathbf{b}) = C(\pi/4) = -1/\sqrt{2} \\ &C(\mathbf{b},\mathbf{c}) = C(\pi/2) = 0 \\ &C(\mathbf{a},\mathbf{c}) = C(3\pi/4) = 1/\sqrt{2} \\ &1 + C(\mathbf{b},\mathbf{c}) < |C(\mathbf{a},\mathbf{b}) - C(\mathbf{a},\mathbf{c})| \end{aligned} \tag{191}$$

The key assumption of Bell's Theorem is that the correlation is computed by multiplying the theoretical measurement results, *A* and *B*, in the integral. This assumes that for a given set of parameters, the measurement result essentially propagates to the detector along with the particle.

In fact, however, it is the spinor wave function ψ which propagates from place to place since the first order Dirac equation is a kind of convection equation. Therefore Bell's Theorem does not generally apply to classical waves.

To compute the correlation between two bispinor wave functions, consider the following properties: First, the magnitude of the wave function must be second-order in each of the components and positive-definite. Therefore:

$$\|\psi\| = \psi^\dagger \psi \tag{192}$$

Second, physical variables are bilinear in the wave function. Therefore it is the squared magnitude which is of physical interest. The un-normalized correlation C_0 between two functions must be defined in such a way that the squared norm $\left|\psi^\dagger \psi\right|^2$ is the self-correlation:

$$C_0(\psi_A, \psi_B) = \left|\psi_A^\dagger \psi_B\right|^2 \tag{193}$$

Dividing by the magnitudes of each wave function yields the normalized correlation C:

$$C(\psi_A, \psi_B) = \frac{C_0(\psi_A, \psi_B)}{\left|\psi_A^\dagger \psi_A\right\|\left\|\psi_B^\dagger \psi_B\right|} = \frac{\left|\psi_A^\dagger \psi_B\right|^2}{\left|\psi_A^\dagger \psi_A\right\|\left\|\psi_B^\dagger \psi_B\right|} \tag{194}$$

The correlation between states related by rotation $R(\boldsymbol{\varphi})$ about an axis perpendicular to the spin is:

$$C = \frac{\left|\psi^\dagger R(\boldsymbol{\varphi})\psi\right|^2}{\left|\psi^\dagger \psi\right|^2} = \cos^2\frac{\varphi}{2} \tag{195}$$

The correlation for angle $(\pi\hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi})$ is $\cos^2[(\pi-\varphi)/2] = \sin^2[\varphi/2]$.

Assuming that spin measurements are coincident or anti-coincident in proportion to the correlations between the spinor wave functions, the correlation C_s between spin measurements separated by angle $\boldsymbol{\varphi}$ is:

$$C_s(\varphi) = C_\psi(\varphi) - C_\psi(\pi - \varphi) = \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2} = \cos\varphi \tag{196}$$

In the case of pair production in EPR-type experiments, the spins of the two electrons (or electron and positron) are opposite (changing φ to $\pi-\varphi$ above), thereby changing the sign of the correlation. Hence the classical correlations are in agreement with the quantum correlations.

3.7.8. Quantum Mechanics

In the preceding section we computed the correlation between two states related by rotation. The two states may be denoted by $\psi(\mathbf{r},t)$ and $R(\boldsymbol{\varphi})\psi(\mathbf{r},t)$. The correlation at a given position and time is given by (195). A more global correlation between two wave functions $\psi_1(\mathbf{r},t)$ and $\psi_2(\mathbf{r},t)$ at a given time is obtained by integrating over space:

$$C = \frac{\left|\int \psi_1^\dagger \psi_2 \, d^3r\right|^2}{\left|\int \psi_1^\dagger \psi_1 \, d^3r\right| \left|\int \psi_2^\dagger \psi_2 \, d^3r\right|} = \frac{\left|\int \psi_2^\dagger \psi_1 \, d^3r\right|^2}{\left|\int \psi_1^\dagger \psi_1 \, d^3r\right| \left|\int \psi_2^\dagger \psi_2 \, d^3r\right|} \tag{197}$$

The correlation between spin one-half states is non-negative, and the correlation of a wave function with itself is unity. These properties provide the basis for a probabilistic interpretation of the wave functions. A given wave function may be decomposed into multiple wave functions (states), and the correlation between the wave function and each 'state' may be computed. In quantum mechanics, this correlation is interpreted as the probability of detecting that state with a measurement.

This means that correlations between physical states (as opposed to measurements) are equal to the square of a complex amplitude. This fundamental property of quantum mechanics has mystified generations of physicists. Yet we can now see clearly that this property of matter is due to the simple fact that independent wave states are 180 degrees apart.

Temporal evolution of the wave function is expressed as:

$$\psi(\mathbf{r},t_2) = \exp\left(-\mathrm{i}\int_{t_1}^{t_2} H(\mathbf{r},t)dt\right)\psi(\mathbf{r},t_1) \tag{198}$$

Therefore the correlation between an initial state $\psi_1(\mathbf{r},t_1)$ and a final state $\psi_2(\mathbf{r},t_2)$ is:

$$C(\psi_2(t_2)\,|\,\psi_1(t_1)) = \frac{\left|\int \psi_2^\dagger \exp\left(-\mathrm{i}\int_{t_1}^{t_2} H(\mathbf{r},t)dt\right)\psi_1 \, d^3r\right|^2}{\left|\int \psi_1^\dagger \psi_1 \, d^3r\right| \left|\int \psi_2^\dagger \psi_2 \, d^3r\right|} \tag{199}$$

In quantum mechanics, the states are normalized to one:

$$\psi' = \frac{\psi}{\left|\int \psi^\dagger \psi \, d^3r\right|^{1/2}} \tag{200}$$

Dropping the primes, the correlation integrals are then written in the form:

$$C(\psi_2(t_2)\,|\,\psi_1(t_1)) \equiv \left|\langle\psi_2\,\|\,\psi_1\rangle\right|^2 = \left|\langle\psi_1\,|\exp\left(-\mathrm{i}\int_{t_1}^{t_2} H(\mathbf{r},t)\,dt\right)\|\psi_1\rangle\right|^2 \tag{201}$$

In quantum mechanics, this correlation represents the probability density for the initial state ψ_1 to evolve into the final state ψ_2.

3.7.9. Fermions and Bosons

Particles whose correlations are computed according to the above rules are called "fermions" in honor of the physicist Enrico Fermi. Fermions are considered to be the "fundamental particles" of nature. These include electrons, protons, neutrons, neutrinos, and quarks. Recall that the Pauli exclusion principle was derived from the assumption that the particle wave functions were eigenfunctions of an observable (e.g. spin). If this is not the case, then there is no exclusion principle.

Particles which can be superposed are called "bosons" in honor of physicist Satyendranath Bose. Examples include photons and π mesons. Multiple bosons may coexist with each in exactly the same state (and same position). In quantum mechanics the boson two-particle wave function satisfies:

$$\psi_A^\dagger\psi_B - \psi_B^\dagger\psi_A = 0 \tag{202}$$

This condition is always satisfied if $\psi_A = \psi_B$, so there is no exclusion principle.

To see how spin is related to statistics, consider a massless photon which in the plane wave approximation satisfies the equation:

$$\partial_t^2\mathbf{Q} - c^2\nabla^2\mathbf{Q} = \left(\partial_t - c\hat{\mathbf{k}}\cdot\nabla\right)\left(\partial_t + c\hat{\mathbf{k}}\cdot\nabla\right)\mathbf{Q} \tag{203}$$

Either $\left(\partial_t + c\hat{\mathbf{k}}\cdot\nabla\right)\mathbf{Q} = 0$ or $\left(\partial_t - c\hat{\mathbf{k}}\cdot\nabla\right)\mathbf{Q} = 0$. In either case the vector **Q** obeys a convection equation and is therefore the quantity used to compute correlations. **Q** is a vector, which transforms under rotation with spin one. Multiple photons can be superposed simply by adding their **Q** values without the interference associated with spinors.

For another example, suppose fermions *A* and *B* are somehow bound together with a joint wave function $\psi_{A,B}$ which satisfies the exclusion principle:

$$\psi_{A,B} = \frac{\psi_B^\dagger\psi_A - \psi_A^\dagger\psi_B}{\sqrt{2}} \tag{204}$$

If we use $\psi_{A,B}$ to compute correlations with an identical particle composed of fermions A' and B', we have (noting that the order of column vectors is irrelevant since they may be regarded as diagonal square matrices):

$$\begin{aligned}&\psi^{\dagger}_{A,B}\psi_{A',B'} - \psi^{\dagger}_{A',B'}\psi_{A,B} \\ &= \left[\frac{\psi^{\dagger}_{A}\psi_{B} - \psi^{\dagger}_{B}\psi_{A}}{\sqrt{2}}\right]\left[\frac{\psi^{\dagger}_{B'}\psi_{A'} - \psi^{\dagger}_{A'}\psi_{B'}}{\sqrt{2}}\right] - \left[\frac{\psi^{\dagger}_{A'}\psi_{B'} - \psi^{\dagger}_{B'}\psi_{A'}}{\sqrt{2}}\right]\left[\frac{\psi^{\dagger}_{B}\psi_{A} - \psi^{\dagger}_{A}\psi_{B}}{\sqrt{2}}\right] = 0\end{aligned} \tag{205}$$

Hence composite particles formed from two fermions behave statistically like bosons.

In the Standard Model of Physics, the viewpoint is that fundamental particles are fermions which interact through fields, and particles associated with the fields are bosons. We have seen how this interpretation can arise from an underlying classical wave process.

3.7.10. Prior Knowledge and Statistics

Interpretation of quantum statistics can be confusing. Consider the case of Schrodinger's cat in Figure[]. The cat is place in a box which contains a radioactive element, a radiation detector, and a poisonous gas. If the detector is triggered by a radioactive decay then it will in turn trigger the release of the poison and thereby kill the cat. According to quantum statistics, at any given time there is not merely a chance that the cat will be dead or alive, but the mathematical description involves a complex amplitude for each possibility. Just as electron statistics were described above by a complex superposition of 'spin up' and 'spin down' states, the cat's fate is described by a complex superposition of 'alive' and 'dead' states. Physicists are therefore tempted to say that the cat is in a superposition of living and dead states, which is rather absurd.

There are different ways to resolve this paradox, but the simplest resolution is to say that the cat really is either dead or alive, and not both. The complex amplitude merely indicates our knowledge (or lack of knowledge) of the situation. Physicists previously rejected this logic because they never realized that classical statistics (e.g. the probability that the cat is dead) should be computed in exactly the same manner as the quantum statistics. The Copenhagen interpretation of quantum mechanics posits that the statistical interpretation of the complex wave function is also the physical interpretation (i.e. there are no deterministic physical variables because if there were then their correlations would be computed differently). However, we can obtain the same correlations without the bizarre interpretation that the cat is partly alive and partly dead until we open the box.

3.7.11. Hydrogen Atom

The proton produces a Coulomb potential ($e\Phi = -Ze^2/r$). Neglecting the vector potential in the electromagnetic electron equation (168) yields:

$$\left[\partial_t + \mathrm{i}\frac{e}{\hbar}\Phi + c\beta_1\boldsymbol{\sigma}\cdot\nabla\right]\psi = -\mathrm{i}\,\Omega\beta_3\psi \tag{206}$$

Assume as before a temporal eigenvalue $\partial_t\psi = -\mathrm{i}E\psi$, and assume that the angular eigenfunction $\Phi_{l,m}^{(+)}$ has even parity and $\Phi_{l,m}^{(-)}$ odd parity. A wave function of the form $\psi^{(+)}$ yields the coupled radial equations:

$$\begin{aligned}\left[E - \frac{e}{\hbar}\Phi - \Omega\right]G + c\left[\partial_r + \frac{\kappa}{r}\right]F = 0\\ \left[E - \frac{e}{\hbar}\Phi + \Omega\right]F - c\left[\partial_r - \frac{\kappa}{r}\right]G = 0\end{aligned} \tag{207}$$

Solutions to these coupled equations are obtained as follows (e.g. Schiff 1968):

For large *r* the asymptotic equations are:

$$\begin{aligned}\left[E - \Omega\right]G + c\partial_r F = 0\\ \left[E + \Omega\right]F - c\partial_r G = 0\end{aligned} \tag{208}$$

which combine to yield:

$$\left[E^2 - \Omega^2\right]F + c^2\partial_r^2 F = 0 \tag{209}$$

We are seeking a bound state with $E^2 < \Omega^2$. Therefore the asymptotic behavior is $F \propto \exp(-\alpha r)$ with $\alpha \equiv \sqrt{\left[\Omega^2 - E^2\right]/c^2}$.

Now let:

$$\begin{aligned}F(r) &= f(r)\exp(-\alpha r)\\ G(r) &= g(r)\exp(-\alpha r)\end{aligned} \tag{210}$$

The coupled equations become:

$$\begin{aligned}\left[E - \frac{e}{\hbar}\Phi - \Omega\right]g + c\left[\partial_r - \alpha + \frac{\kappa}{r}\right]f = 0\\ \left[E - \frac{e}{\hbar}\Phi + \Omega\right]f - c\left[\partial_r - \alpha - \frac{\kappa}{r}\right]g = 0\end{aligned} \tag{211}$$

Assume that *f* and *g* can be written as power series:

$$f(r) = \sum_{\nu=0}^{\infty} f_\nu r^{s+\nu}$$
$$g(r) = \sum_{\nu=0}^{\infty} g_\nu r^{s+\nu} \tag{212}$$

Let $\gamma \equiv Ze^2/\hbar c$ and match powers of r:

$$\begin{aligned} [E-\Omega]g_{\nu-1} - c\alpha f_{\nu-1} + \gamma c g_\nu + c[s+\nu+\kappa]f_\nu = 0 \\ [E+\Omega]f_{\nu-1} + c\alpha g_{\nu-1} + \gamma c f_\nu - c[s+\nu-\kappa]g_\nu = 0 \end{aligned} \tag{213}$$

We can eliminate the $\nu-1$ terms to get a relationship between f_ν and g_ν:

$$\{[E-\Omega]c[s+\nu-\kappa] + c^2\alpha\gamma\}g_\nu = \{[E-\Omega]\gamma c - c^2\alpha[s+\nu+\kappa]\}f_\nu \tag{214}$$

which for large ν becomes $[E-\Omega]g_\nu = -c\alpha f_\nu$.

For ν=0:

$$\begin{aligned} \gamma g_0 + [s+\kappa]f_0 = 0 \\ \gamma f_0 - [s-\kappa]g_0 = 0 \end{aligned} \tag{215}$$

The determinant for these coupled equations must be zero. This condition yields a solution for s:

$$s = \pm\sqrt{\kappa^2 - \gamma^2} \tag{216}$$

Recall that the actual wave function contains an additional factor of $1/r$. Therefore we choose the positive sign here so that the solution is regular (or only slightly divergent if $|s|<1$) at the origin.

Using the relation between coefficients derived above, the asymptotic behavior for large ν is:

$$\begin{aligned} -2\alpha f_{\nu-1} + \nu f_\nu = 0 \\ 2\alpha g_{\nu-1} - \nu g_\nu = 0 \end{aligned} \tag{217}$$

The ratio between successive terms matches the Taylor series expansion for $\exp(2\alpha r)$:

$$\exp(2\alpha r) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}[2\alpha r]^\nu \tag{218}$$

If the series proceeds to infinite ν then the wave function would be infinite at large values of *r*. To make the wave function finite, the series must terminate at some finite value of ν. Calling this value *n*', Eq. (213) yields the relation between the highest coefficients:

$$c\alpha g_{n'} = -[E+\Omega] f_{n'} \tag{219}$$

Combining this relation with eq. (214) yields an expression for the characteristic frequencies:

$$E\gamma = c^2[s+n']\alpha = c^2[s+n']\left[\Omega^2 - E^2\right]^{1/2} \tag{220}$$

Solving for $\hbar E$:

$$\hbar E = \hbar\Omega\left\{1+\gamma^2/[s+n']^2\right\}^{-1/2} \tag{221}$$

These are the discrete energy levels of an electron in a Coulomb potential. The factor of $\hbar$, which relates energy and frequency, is assumed to be the integral of the squared wave function. Denote the energy by $\mathcal{E} \equiv \hbar E$ and mass by $m_e c^2 \equiv \hbar\Omega$. These energy levels were actually derived by Sommerfeld [1916a] using the model of a relativistic particle propagating in elliptical orbits.

There are two main sources of discrepancy from the actual hydrogen energy levels. First, we assumed a static potential, implying that the nucleus is unaffected by the presence of the electron wave. By analogy with particles we can improve the calculations by replacing the electron mass $\hbar\Omega = m_e c^2$ with the "reduced mass" $\hbar\Omega' = c^2 m_e m_p / [m_e + m_p]$, where m_p is the proton mass. Second, we have neglected any effects of the magnetic vector potential.

The energy levels are typically classified using a positive integer principal quantum number *n* and positive half-integer angular quantum number $J \equiv \kappa - 1/2$:

$$n = J + \frac{1}{2} + n' \tag{222}$$

In terms of these quantum numbers the energy levels are:

$$\mathcal{E} = \hbar\Omega'\left\{1+\frac{\gamma^2}{\left[n-J-1/2+\sqrt{[J+1/2]^2-\gamma^2}\right]^2}\right\}^{-1/2} \tag{223}$$

The table below compares measured energy levels (relative to the ground state) with energy levels calculated using this formula. The configuration label (*nL*) includes the principal quantum number *n* followed by a letter code for the orbital angular momentum *L*: s=0, p=1, d=2, f=3, etc.

Notice that the formula above does not distinguish between different L values for the same n and J.

Configuration	J	Measured Level (eV)	Level Computed from (221)
1s	1/2	0	0
2s	1/2	10.1988101	10.1988390
2p	1/2	10.1988057	10.1988390
2p	3/2	10.1988511	10.1988843
3s	1/2	12.0874944	12.0875263
3p	1/2	12.0874931	12.0875263
3p	3/2	12.0875066	12.0875397
3d	3/2	12.0875065	12.0875397
3d	5/2	12.0875110	12.0875442
4s	1/2	12.7485324	12.7485650
4p	1/2	12.7485319	12.7485650
4p	3/2	12.7485375	12.7485707
4d	3/2	12.7485375	12.7485707
4d	5/2	12.7485394	12.7485726
4f	5/2	12.7485394	12.7485726
4f	7/2	12.7485404	12.7485735
$n\rightarrow\infty$		13.5984340	13.5984671

Table II. Measured and computed hydrogen energy levels.

Ralchenko, Yu., Jou, F.-C., Kelleher, D.E., Kramida, A.E., Musgrove, A., Reader, J., Wiese, W.L., and Olsen, K. (2007). NIST Atomic Spectra Database (version 3.1.2), [Online]. Available: http://physics.nist.gov/asd3 [2007, May 8]. National Institute of Standards and Technology, Gaithersburg, MD.

While the agreement with experiment is good, it must be noted that the assumed Coulomb potential is simply empirical (as it is also in conventional quantum theory). For a complete theory the potentials of the nucleus should be derived from its free particle wave function.

3.8. Symmetries

"I cannot believe that God is a weak left-hander…"

— Wolfgang Pauli

3.8.1. Spatial inversion

Spatial inversion (conventionally called the parity operation, *P*, though we will use the letter *M* for mirroring) is the process of inverting the three spatial axes. This operation corresponds to a mirror image followed by a 180 degree rotation about the axis perpendicular to the mirror. Since rotation does not affect any physical laws, we will sometimes substitute the term "mirror image" for "spatial inversion" when referring to general physical consequences. Parity conservation is generally taken to mean that when spatial inversion is applied to any physical process, the resulting process is equally frequent in nature. Parity violation means that a process and its mirror image are not equally likely, and maximal parity violation means that spatial inversion of a physical process yields a process with no physical interpretation.

In this chapter, we are not interested in the relative frequency of occurrence of events and their mirror images. We are only concerned with the question of maximal parity violation: "Is the mirror image process possible in nature or not?" We will refer to maximal parity violation as "mirror asymmetry", and existence of a mirror image process as "mirror symmetry."

When viewed in a mirror, all known physical processes appear to proceed as if matter and anti-matter were exchanged. The simplest explanation for this observation is that spatial inversion exchanges matter and anti-matter. Let us consider how the wave function changes under spatial inversion.

3.8.1.1. Conventional parity operator

Dirac's original equation for a free particle has the form:

$$\partial_t \psi + c\beta_1 \sigma_i \partial_i \psi = -\mathrm{i}\Omega\beta_3 \psi \tag{224}$$

where $\Omega \equiv mc^2/\hbar$. The β-matrices may be taken as:

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \qquad \beta_2 = \begin{pmatrix} 0 & 0 & -\tilde{\mathrm{i}} & 0 \\ 0 & 0 & 0 & -\tilde{\mathrm{i}} \\ \tilde{\mathrm{i}} & 0 & 0 & 0 \\ 0 & \tilde{\mathrm{i}} & 0 & 0 \end{pmatrix}; \qquad \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{225}$$

Where $\tilde{\mathrm{i}}$ is the pseudoscalar imaginary, as will be seen below.

The spin matrices σ^i utilize a true scalar imaginary ($\bar{\mathrm{i}}$):

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -\bar{\mathrm{i}} & 0 & 0 \\ \bar{\mathrm{i}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\mathrm{i}} \\ 0 & 0 & \bar{\mathrm{i}} & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{226}$$

Multiplying the Dirac equation by $\psi^\dagger$ and adding the Hermitian conjugate equation yields a continuity equation:

$$\partial_t\left[\psi^\dagger\psi\right]+\nabla\cdot\left[\psi^\dagger\beta_1\boldsymbol{\sigma}\psi\right]=0 \tag{227}$$

This relationship is sufficient to establish the probability density ($\psi^\dagger\psi$) and current ($\psi^\dagger\beta_1\boldsymbol{\sigma}\psi$) as the components of a Lorentz four-vector.

Although the above analysis is satisfactory, it is currently fashionable to use the notation:

$$\gamma^0\equiv\beta_3;\ \gamma^5\equiv\beta_1;\ \gamma^i\equiv\gamma^0\gamma^5\sigma_i \tag{228}$$

and multiply each term in the original Dirac equation (224) by γ^0 to obtain:

$$\gamma^0\partial_t\psi+c\gamma^i\partial_i\psi\equiv\gamma^\mu\partial_\mu\psi=-\mathrm{i}\Omega\psi \tag{229}$$

This procedure cannot have any effect on the transformation properties of the Dirac matrices.

The conventional parity operator *P* is assumed to have the form: $P\psi(\mathbf{r})=U\psi(-\mathbf{r})$. It is derived from the requirement that the Dirac equation in the form (229) be invariant with respect to the transformation:

$$\gamma^0\partial_t U\psi(-\mathbf{r})+c\gamma^i\partial_i U\psi(-\mathbf{r})+\mathrm{i}\Omega U\psi(-\mathbf{r})=0 \tag{230}$$

Inverting the parity operator yields:

$$U^{-1}\gamma^0U\partial_t\psi(\mathbf{r})-cU^{-1}\gamma^iU\partial_i\psi(\mathbf{r})+\mathrm{i}\Omega U^{-1}U\psi(\mathbf{r})=0 \tag{231}$$

Equivalence with the original Dirac equation requires:

$$\begin{aligned}&U^{-1}\gamma^0U=\gamma^0\\&U^{-1}\gamma^iU=-\gamma^i\\&U^{-1}U=1\end{aligned} \tag{232}$$

These conditions are satisfied by $U=\gamma^0$. Within an arbitrary phase factor the conventional parity operator is therefore:

$$P\psi(\mathbf{r})=\gamma^0\psi(-\mathbf{r})=\beta_3\psi(-\mathbf{r}) \tag{233}$$

There are two problems with this derivation. First, the form $P\psi(\mathbf{r})=U\psi(-\mathbf{r})$ is not the most general possible operator. For example, the conventional charge conjugation operator includes

complex conjugation. Second, the matrix $\gamma^0 = \beta_3$ is not inverted because it is presumed to represent a temporal component of a four-vector. This illusion is maintained by rewriting the probability density and current components as $\overline{\psi}\gamma^0\psi$ and $\overline{\psi}\gamma^i\psi$, respectively, with $\overline{\psi} \equiv \psi^\dagger\gamma^0$. This change of notation does not change the fact, however, that the probability density is independent of γ^0. The matrix associated with the temporal part of the probability current 4-vector is the identity matrix, not γ^0. This is an important flaw in the conventional derivation of the parity operator.

Since the 4-vector $(\psi^\dagger\psi, \psi^\dagger\gamma^5\boldsymbol{\sigma}\psi)$ is indeed Lorentz-invariant, there is absolutely no basis for the claim that γ^0 is a temporal component. On the contrary, we will show that γ^0 is geometrically related to wave velocity and may quite reasonably be inverted by spatial inversion. We will see that the resulting spatial inversion operator inverts all of the terms in the modified Dirac equation (229).

3.8.1.2. New spatial inversion operator

In discussing spatial inversion, it will be necessary to define two different unit imaginary numbers. As defined above, the product of spin matrices is a true scalar with respect to spatial inversion:

$$\bar{\mathrm{i}} \equiv \sigma^1\sigma^2\sigma^3 \tag{234}$$

The σ-matrices are not involved in spatial inversion, which inverts the wave velocity but not the spin. However, we can identify three matrices associated with polar vectors which have the same algebra as the σ-matrices.

The β matrices define directions relative to the velocity vector $\langle c\beta_1\boldsymbol{\sigma}\rangle = \langle c\gamma^5\boldsymbol{\sigma}\rangle$, where the brackets indicate expectation value. One can also define absolute vectors $(\langle\beta_1\boldsymbol{\sigma}\rangle, \langle\beta_2\boldsymbol{\sigma}\rangle, \langle\beta_3\boldsymbol{\sigma}\rangle)$.[1] If the wave function is an eigenfunction of velocity aligned with a spatial axis x_v so that $c\beta_1\sigma_v\psi = c\psi$, then (using $\sigma_v^2 = 1$):

$$\begin{aligned}
\psi^\dagger c\beta_3\sigma_v\psi &= \left[\psi^\dagger\beta_1\sigma_v\right]c\beta_3\sigma_v\left[\beta_1\sigma_v\psi\right] = -\psi^\dagger c\beta_3\sigma_v\psi = 0 \\
\psi^\dagger c\beta_2\sigma_v\psi &= \left[\psi^\dagger\beta_1\sigma_v\right]c\beta_2\sigma_v\left[\beta_1\sigma_v\psi\right] = -\psi^\dagger c\beta_2\sigma_v\psi = 0
\end{aligned} \tag{235}$$

These results follow from the fact that β_1 is a reflection operator for both β_2 and β_3, and the only number equal to its negative is zero. Therefore $\langle\beta_2\boldsymbol{\sigma}\rangle$ and $\langle\beta_3\boldsymbol{\sigma}\rangle$ are indeed perpendicular to velocity $\langle\beta_1\boldsymbol{\sigma}\rangle$ for velocity eigenfunctions. For example, in our notation the wave function $\psi_1 = (1 \quad 0 \quad 0 \quad 1)^T$ is a simultaneous eigenfunction of $\beta_1\sigma_1$, $\beta_2\sigma_2$, and $\beta_3\sigma_3$. Therefore the

three vectors $\langle\beta_1\boldsymbol{\sigma}\rangle$, $\langle\beta_2\boldsymbol{\sigma}\rangle$, and $\langle\beta_3\boldsymbol{\sigma}\rangle$ are mutually orthogonal vectors (left-handed) in three dimensional space, at least for velocity eigenfunctions. The vector $\langle\beta_1\boldsymbol{\sigma}\rangle$ is parallel to $\hat{\mathbf{x}}_1$. Rotation of the vector $\langle\beta_1\boldsymbol{\sigma}\rangle$ by –90 degrees about the relative vector β_2 yields $\langle\beta_3\boldsymbol{\sigma}\rangle$, which is parallel to $\hat{\mathbf{x}}_3$. This is of course the same as rotation of $\hat{\mathbf{x}}_1$ by –90 degrees about $\hat{\mathbf{x}}_2$, which is associated with the matrix σ_2. It is therefore clear that for velocity eigenfunctions, the relative vectors represented by $(\beta_1,\beta_2,\beta_3)$ are geometrically equivalent to the absolute vectors represented by $(\sigma_1,\sigma_2,\sigma_3)$. We assume that all three vectors $\langle\beta_1\boldsymbol{\sigma}\rangle$, $\langle\beta_2\boldsymbol{\sigma}\rangle$, and $\langle\beta_3\boldsymbol{\sigma}\rangle$ are polar vectors so that the vector space $(\beta_1,\beta_2,\beta_3)$ does not have mixed parity.

The matrix factor $\gamma^0=\beta_3$ in the conventional parity operator represents a rotation by 180 degrees about the β_3 axis ($\hat{\mathbf{x}}_3$ in our example). This operation inverts only two of the three orthogonal vectors associated with velocity.

Compare this situation with classical transverse waves in a solid. We could define an operator (analogous to the Dirac *P* operator) which reflects the equilibrium position of each point in the solid, and also reflects the wave velocity direction. We also invert local displacements and velocities along one of the two axes perpendicular to the wave velocity. The resulting "reflected" wave would propagate along just as one would expect for the spatially inverted wave. But of course the operator we defined is not the spatial inversion operator, because we failed to invert one of the axes of the local displacement and velocity of the solid medium (in total we inverted two of the three local axes, corresponding to a 180 degree rotation about the third axis). Similarly, the Dirac *P* operator inverts the "wave" (or "particle") velocity direction, but inverts only one of two other quantities which are geometrically related to the "wave" velocity (a 180 degree rotation in the velocity-representation space). We will derive a new spatial inversion operator which inverts all three vectors $\langle\beta_1\boldsymbol{\sigma}\rangle$, $\langle\beta_2\boldsymbol{\sigma}\rangle$, and $\langle\beta_3\boldsymbol{\sigma}\rangle$ associated with velocity.

The spin matrices σ_i are components of a pseudovector and should not be inverted. Therefore the spatial inversion must be accomplished by inverting the three relative matrices $(\beta_1,\beta_2,\beta_3)$. This requires that the associated imaginary $\tilde{\mathrm{i}}$ be a pseudoscalar, as assumed above. The unit imaginary associated with mass is assumed to be a pseudoscalar since it is multiplied by (β_3) in the original Dirac equation.

The roles of the different imaginaries can be clarified by factoring the Dirac wave function in a manner similar to that of Hestenes [1967]:

$$\psi(\mathbf{r})\equiv a^{1/2}\exp\left(\bar{\mathrm{i}}\,\sigma_i\varphi_i\right)\exp\left(\beta_1\sigma_i\alpha_i\right)\exp\left(\tilde{\mathrm{i}}\,\beta_1\zeta\right)\psi_0 \tag{236}$$

It is clear that $\bar{\mathrm{i}}\,\sigma_i=\varepsilon_{ijk}\sigma_j\sigma_k/2$ is associated with rotation in the plane orthogonal to the x_i axis. Similarly, $\tilde{\mathrm{i}}\,\beta_1=\beta_2\beta_3$ is associated with rotation in the velocity-representation space.

Next we define a new wave function in which all imaginary pseudoscalar factors are inverted: $\psi^{\#}\left(\tilde{\mathrm{i}}\right)\equiv\psi\left(-\tilde{\mathrm{i}}\right)$. This pseudoscalar conjugation operation differs from complex conjugation, which inverts both scalar and pseudoscalar imaginaries. Pseudoscalar conjugation inverts $\langle\beta_2\rangle$ since:

$$\psi^{\#\dagger}\beta_2\psi^{\#}=\left[\psi^{\dagger}\beta_2{}^{\#}\psi\right]^{\#}=\left[\psi^{\dagger}\left[-\beta_2\right]\psi\right]^{\#}=\psi^{\dagger}\left[-\beta_2\right]\psi \tag{237}$$

The spatial inversion (or mirroring operator *M*) which inverts all of the relative velocity vectors, is then (within an arbitrary phase factor):

$$M\psi(\mathbf{r})\equiv\psi_M(\mathbf{r})=\beta_2\psi^{\#}(-\mathbf{r}) \tag{238}$$

This operator inverts observables computed from β_1, β_2, and β_3 independently of the change in sign of **r**.

The Dirac equation for a particle in electromagnetic potentials is:

$$\left[\partial_t+c\beta_1\sigma^i\partial_i+\tilde{\mathrm{i}}\beta_3\Omega+\mathrm{i}e\Phi-\mathrm{i}e\beta_1\sigma^i\mathbf{A}_i\right]\psi=0 \tag{239}$$

When applied to this equation, the parity operator inverts β_3, β_1, $\tilde{\mathrm{i}}$, and ∂_i (the matrices are inverted because they anti-commute with β_2). Denoting spatially inverted quantities with subscript *M*, the spatially inverted Dirac equation is:

$$\left[\partial_t+c\beta_1\sigma^i\partial_i+\tilde{\mathrm{i}}\beta_3\Omega_M+\mathrm{i}^{\#}e_M\Phi_M+\mathrm{i}^{\#}\beta_1\sigma^i\left(e_M\mathbf{A}_{Mi}\right)\right]\psi_M=0 \tag{240}$$

We assume $\Omega_M=\Omega$. The transformed equation has the same form as the original Dirac equation except for the sign of the vector potential term. This sign change is necessary for consistency with gauge transformations. The gauge transformation

$$\begin{aligned}e\Phi'&=e\Phi+\partial_t\chi\\ eA_i'&=eA_i-\nabla\chi\\ \psi'&=\exp\left(-\mathrm{i}\chi\right)\end{aligned} \tag{241}$$

suggests that the scalar potential may be regarded as a time derivative and the vector potential may be regarded as a spatial derivative. Taking $\Phi\equiv\partial_t g$ and $\mathbf{A}\equiv\nabla\times\mathbf{G}+\nabla g$ would leave the form of the equation invariant:

$$\left[\partial_t+c\beta_1\sigma^i\partial_i+\tilde{\mathrm{i}}\beta_3\Omega_M+\mathrm{i}^{\#}e_M\partial_t g_M-\mathrm{i}^{\#}e_M\beta_1\boldsymbol{\sigma}\cdot\left(\nabla\times\mathbf{G}_M+\nabla g_M\right)\right]\psi_M=0 \tag{242}$$

The scalar and vector potentials must have opposite spatial inversion eigenvalues. We will assume that:

$$\begin{aligned} M\left[\mathrm{i}e\Phi(\mathbf{r})\right] &= \mathrm{i}^{\#} e_M \Phi_M = -\mathrm{i}e\Phi(-\mathbf{r}) \\ M\left[\mathrm{i}e\mathbf{A}(\mathbf{r})\right] &= \mathrm{i}^{\#} e_M \mathbf{A}_M = \mathrm{i}e\mathbf{A}(-\mathbf{r}) \end{aligned} \tag{243}$$

The transformed Dirac equation is then:

$$\left[\partial_t + c\beta_1\sigma^i\partial_i + \tilde{\mathrm{i}}\beta_3\Omega_M - \mathrm{i}e_M\Phi(-\mathbf{r}) - \mathrm{i}e_M\beta_1\sigma_i A_i(-\mathbf{r})\right]\psi_M = 0$$

With these transformation properties, we will show that the new parity operator is consistent with an exchange of matter and anti-matter.

3.8.1.3. Eigenfunctions and eigenvalues

Next we consider the effect of the new parity operator on the eigenvalue equation. For simplicity we assume the vector potential **A** to be zero. Assuming temporal dependence $\exp(-\mathrm{i}Et)$, the eigenvalue equation is:

$$\left[-\mathrm{i}E + \mathrm{i}e\Phi + c\beta_1\boldsymbol{\sigma}\cdot\nabla\right]\psi = -\tilde{\mathrm{i}}\,\Omega\beta_3\psi \tag{244}$$

The operator $\boldsymbol{\sigma}\cdot\nabla\psi$ can be factored:

$$\boldsymbol{\sigma}\cdot\nabla\psi = \sigma_r\left[\partial_r + \mathrm{i}\frac{\boldsymbol{\sigma}}{r}\cdot\left[\mathbf{r}\times\nabla\right]\right]\psi = \sigma_r\left[\partial_r - \frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right]\psi \tag{245}$$

The two-component angular solutions of the eigenvalue equations $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(+)}_{l,m} = l = -1+\kappa$ and $\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(-)}_{l,m} = -[l+2] = -1-\kappa$ are well known.[2] These two angular solutions are related by $\sigma_r\Phi^{(+)}_{l,m} = \Phi^{(-)}_{l,m}$ and yield opposite eigenvalues under coordinate inversion ($\mathbf{r}\to\mathbf{r}$). Only the true scalar imaginary $\bar{\mathrm{i}}$ can appear within these functions.

Denote two wave functions as:

$$\psi^{(+)} = \frac{1}{r}\begin{bmatrix} \tilde{\mathrm{i}}\,G\Phi^{(+)}_{l,m} \\ F\Phi^{(-)}_{l,m} \end{bmatrix} \quad \text{or} \quad \psi^{(-)} = \frac{1}{r}\begin{bmatrix} \tilde{\mathrm{i}}\,F\Phi^{(-)}_{l,m} \\ G\Phi^{(+)}_{l,m} \end{bmatrix} \tag{246}$$

Each of these is an eigenfunction of the conventional parity operator, but they are exchanged by the new spatial inversion operator:

$$M\psi^{(+)}(\mathbf{r}) = \gamma^4\psi^{(+)\#}(-\mathbf{r}) = (-)^l\psi^{(-)}(\mathbf{r})$$
$$M\psi^{(-)}(\mathbf{r}) = \gamma^4\psi^{(-)\#}(-\mathbf{r}) = (-)^{l+1}\psi^{(+)}(\mathbf{r}) \tag{247}$$

Using $\psi^{(+)}$ in the (original) Dirac equation yields the coupled radial equations:

$$\left[E - e\Phi - \Omega\right]G + c\left[\partial_r + \frac{\kappa}{r}\right]F = 0$$
$$\left[E - e\Phi + \Omega\right]F - c\left[\partial_r - \frac{\kappa}{r}\right]G = 0 \tag{248}$$

$\psi^{(-)}$ yields similar coupled equations with opposite sign of E and $e\Phi$, as expected for exchange of matter and anti-matter. The energy levels for $\psi^{(+)}$ in a negative Coulomb potential are therefore equal and opposite to the energy levels of $\psi^{(-)}$ in a positive Coulomb potential. The need for this result was the reason for assuming that the parity operator locally inverts the scalar potential term $\mathrm{i}e\Phi(\mathbf{r})$.

3.8.1.4. Weak interactions

The projection operator for left-handed spinor components is:

$$\psi_L = (I - \beta_1)\psi \tag{249}$$

The unit matrix I is a scalar and β_1 is a pseudoscalar. However, the projection operator does not violate mirror symmetry so long as the reflected counterpart $\psi_R = (I + \beta_1)M\psi$ is as physically plausible as the original projected wave function. Since the new spatial inversion operator exchanges matter and antimatter, all of the elementary particles involved in the weak interaction do in fact have spatially reflected counterparts in nature (electrons and positrons, left-handed neutrinos and right-handed anti-neutrinos, etc.). The mathematical form of the weak vertex factor is entirely consistent with mirror symmetry.

3.8.1.5. Comparison with conventional *PC*

The conventional *PC* operator is:

$$PC\psi(\mathbf{r}) = \mathrm{i}\gamma^0\gamma^2\psi^*(-\mathbf{r}) = \mathrm{i}\gamma^5\sigma_2\psi^*(-\mathbf{r}) \tag{250}$$

This differs from our spatial inversion operator by an arbitrary phase factor, the factor of $\gamma^0\sigma_2$ and conjugation of the scalar imaginary (denoted by $\psi \to \psi^{*\#}$). The factor σ_2 is, within a

phase factor, simply a rotation by π about the x_2 axis: $\sigma_2 = \bar{\mathrm{i}}\exp\left(-\bar{\mathrm{i}}\,\sigma_2\,\pi/2\right)$. Complex conjugation of the scalar imaginary inverts the spin component S_2:

$$PCS_2 = \psi^{*\dagger}\sigma_2\psi^{*} = \left[\psi^{\dagger}\sigma_2{}^{*}\psi\right]^{*} = \left[\psi^{\dagger}\left[-\sigma_2\right]\psi\right]^{*} = -\psi^{\dagger}\sigma_2\psi = -S_2 \tag{251}$$

Therefore the net effect of $\psi \to \sigma^2\psi^{*\#}$ is to invert the spin.

The additional factor of γ^0 inverts velocity by rotation of the velocity-representation matrices. Applied to the matter and anti-matter eigenfunctions, it is equivalent to inverting the spatial arguments in the wave functions. Therefore the conventional *PC* operator, though it exchanges matter and anti-matter, differs significantly from the new spatial inversion operator *M*.

3.8.2. Time reversal

Physically, time reversal must invert the time derivative operator, velocity, and spin independently of the change in argument. One of the electromagnetic potentials must also be inverted. Velocity and spin are both inverted by the transformation:

$$B\psi(t) = \psi_B(t) = \sigma^2\psi^{*\#}(-t) \tag{252}$$

The velocity-representation space $\left(\gamma^5, \gamma^4, \gamma^0\right)$ is unaffected by this transformation. By contrast, the conventional time reversal operator $T\psi(t) = \mathrm{i}\,\sigma^2\psi^{*}(-t)$ inverts γ^4 but not other matrices of velocity-representation space. This suggests that the conventional time reversal operator is also incorrect. However, unlike the conventional parity transformation, there is no empirical evidence to validate this claim.

Applied to the Dirac equation, the new time reversal operator yields:

$$\begin{aligned} &B\left\{\left[\partial_t + c\gamma^5\sigma^i\partial_i + \tilde{\mathrm{i}}\,\gamma^0\Omega + \mathrm{i}e\Phi - \mathrm{i}e\gamma^5\sigma^i A_i\right]\psi\right\} \\ &= -\left[\partial_t + c\gamma^5\sigma^i\partial_i - \tilde{\mathrm{i}}\,\gamma^0\Omega_B - \mathrm{i}^{*\#}e_B\Phi_B - \mathrm{i}^{*\#}\gamma^5\sigma^i e_B A_{Bi}\right]\psi_B = 0 \end{aligned} \tag{253}$$

We recover the original form of the Dirac equation if $\Omega_B = -\Omega$ (i.e. Ω is an eigenvalue of an operator which transforms like a time derivative) and the potentials are interpreted as derivatives.

We assume the potentials to transform as:

$$\begin{aligned} B\left[\mathrm{i}e\Phi(t)\right] &= \mathrm{i}^{*\#}e_B\Phi_B = \mathrm{i}e\Phi(-t) \\ B\left[\mathrm{i}e\mathbf{A}\right] &= \mathrm{i}^{*\#}e_B\mathbf{A}_B = -\mathrm{i}e\mathbf{A}(-t) \end{aligned} \tag{254}$$

According to our interpretation of matter and anti-matter as mirror-images, time reversal does not exchange the two.

3.8.3. Combined Transformations

The combined *MB* transformation is:

$$MB\psi(\mathbf{r},t) = \gamma^4\sigma_2\psi^*(-\mathbf{r},-t) = \mathrm{i}\gamma^2\psi^*(-\mathbf{r},-t) \tag{255}$$

This is closely related to the conventional charge conjugation transformation C:

$$C\psi(\mathbf{r},t) = \gamma^4\sigma_2\psi^*(\mathbf{r},t) = \gamma^2\psi^*(\mathbf{r},t) \tag{256}$$

The conventional charge conjugation operator inverts the spin and velocity in place, without inverting the spatial or temporal coordinates. In terms of dynamical behavior, charge conjugation has the same effect as inverting the sign of the electromagnetic potentials in the Dirac equation.

The conventional *PT* transformation is:

$$PT\psi(\mathbf{r},t) = \gamma^0\sigma_2\psi^*(-\mathbf{r},-t) \tag{257}$$

This differs from the new *MB* transformation by the factor γ^5, which rotates the velocity-representation space by 180 degrees.

The conventional *PCT* transformation is:

$$PCT\psi(\mathbf{r},t) = \gamma^5\psi(-\mathbf{r},-t) \tag{258}$$

This transformation is the conventional theoretical relation between matter and antimatter. Compared with the *MB* operator, it differs only by charge conjugation (which has similar effect to restoring the potentials inverted by *MB*) and by the factor γ^5.

3.9. Mathematical and Physical Properties of Spinors

> "*…our present thinking about quantum mechanics is infested with the deepest misconceptions.*"
>
> — Stephen Gull, Anthony Lasenby, and Chris Doran [1993]

3.9.1. Spinors and Inner Products

An understanding of some mathematical properties of spinors will be useful. Expressions for physical quantities (e.g. Q) are computed from operators (e.g. *Q*) in the form:

$$\mathrm{Q} = \frac{1}{2}\Big[[Q\psi]^{\dagger}\psi + \psi^{\dagger}[Q\psi]\Big] = \frac{1}{2}\Big[\psi^{\dagger}[Q\psi] + \big[\psi^{\dagger}[Q\psi]\big]^{\dagger}\Big] \tag{259}$$

Since the adjoint of a scalar is its complex conjugate, the physical quantity Q is real-valued. When integrated over space, such expressions take the form of an inner product:

$$\langle \mathrm{Q} \rangle = \frac{1}{2}[(f,g) + (g,f)] = \frac{1}{2}\int \Big[f^{\dagger}g + g^{\dagger}f\Big] d^3r \tag{260}$$

The quantity <Q> is the integrated value (or expectation value in QM).

A complete space of functions with an inner product satisfying some simple properties (*e.g.* linearity) is called a 'Hilbert space'. It suffices for our purposes to say that the inner product defined above satisfies all of the necessary criteria.

[Note: the inner product is often defined using only one of the terms in the integrand above (without the factor of one-half). With this definition local densities may be complex even though the integral is real.]

The inner product between two spinor functions is analogous to the dot product between two vectors or the correlation between two scalar functions. The inner product of a spinor function with itself is its positive-definite magnitude:

$$\|f\|^2 = (f,f) = \int f^{\dagger} f \, d^3r \geq 0 \tag{261}$$

In terms of components this is:

$$(f,f) = \int \sum_{\alpha} f_{\alpha}^{*} f_{\alpha} \, d^3r \geq 0 \tag{262}$$

The local projection $p_{\Phi}\psi(\mathbf{r})$ of one function $\psi(\mathbf{r})$ onto another function $\Phi(\mathbf{r})$ is defined as:

$$p_{\Phi}\psi(\mathbf{r}) = \frac{\Phi^{\dagger}(\mathbf{r})\psi(\mathbf{r})}{\Phi^{\dagger}(\mathbf{r})\Phi(\mathbf{r})}\Phi(\mathbf{r}) \tag{263}$$

The global projection $P_{\Phi}\psi(\mathbf{r})$ of one function ψ onto another function Φ is defined as:

$$P_\Phi \psi(\mathbf{r}) = \frac{(\Phi, \psi)}{\|\Phi\|^2} \Phi(\mathbf{r}) \tag{264}$$

The term 'projection' by itself generally refers to the global projection in the literature. For comparison, the projection of a vector **a** onto a vector **b** is the component of **a** that is parallel with **b**:

$$P_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} = [\mathbf{a} \cdot \hat{\mathbf{b}}] \hat{\mathbf{b}} \tag{265}$$

If an operator has Hermitian ($H^\dagger = H$) and anti-Hermitian ($A^\dagger = -A$) parts, then only the Hermitian part contributes to the physical value:

$$\mathrm{Q} = \frac{1}{2}\left[[[H + A]\psi]^\dagger \psi + \psi^\dagger [[H + A]\psi]\right] = \frac{1}{2}\left[\psi^\dagger \left[H^\dagger + A^\dagger + H + A\right]\psi\right] = \psi^\dagger H \psi$$

From this we can conclude that the condition for a real-valued inner product is that the operator is Hermitian ($Q^\dagger = Q$). For example consider the spatial derivative $\nabla_j = \partial / \partial x_j$:

$$(f, \nabla_j f) = \int f^\dagger \nabla_j f \, d^3 r \tag{266}$$

The adjoint is:

$$(f, \nabla_j f)^\dagger = \int [\nabla_j f]^\dagger f \, d^3 r \tag{267}$$

Integration by parts yields:

$$\int [\nabla_j f]^\dagger f \, d^3 r = \int \nabla_j f^\dagger f \, d^3 r = \left[\iint dS_j \; f^\dagger f\right]_{x_{1j}}^{x_{2j}} - \iiint d^3 r \; f^\dagger [\nabla_j f] \tag{268}$$

We assume that the spinor functions fall to zero prior to reaching the boundary of integration (*i.e.* that the boundary is sufficiently far that there is no contribution to the volume integral outside the boundary). This assumption allows us to discard the boundary term, but limits our ability to give physical interpretation to the local functions. Assuming the boundary contribution to be zero, we have:

$$(f, \nabla_j f)^\dagger = -(f, \nabla_j f) \tag{269}$$

Hence the spatial derivative is an anti-Hermitian operator (minus sign rather than plus sign).

Clearly this property holds for all components of the gradient, so we can write:

$$(f, \nabla f)^\dagger = -(f, \nabla f) \tag{270}$$

Which leads to the rather obvious expression for the integrated value:

$$\langle\nabla\rangle = \int [\nabla f]^\dagger f + f^\dagger \nabla f \, d^3r = \int - f^\dagger \nabla f \, d^3r + f^\dagger \nabla f \, d^3r = 0$$

This relationship in operator form is:

$$[\nabla f]^\dagger = [\nabla^\dagger f]^\dagger = -f^\dagger \nabla \tag{271}$$

Note that the form of the gradient operator is not changed by the adjoint operation ($\nabla^\dagger = \nabla$). The sign change comes from transposing the operator from the left to the right side (via integration by parts). Note that:

$$\nabla[f^\dagger f] = [\nabla f^\dagger] f + f^\dagger [\nabla f] \tag{272}$$

This expression is obviously not zero in general, but its volume integral is zero as long as the function *f* falls off sufficiently rapidly near the integration boundaries.

It is simple to construct a Hermitian operator from the gradient operator by multiplying it with the unit imaginary:

$$(f, \mathrm{i}\nabla f)^\dagger = +(f, \mathrm{i}\nabla f) \tag{273}$$

3.9.2. Matrix Algebra

Before proceeding further, it will be useful to tabulate some relationships between matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{274}$$

In the Dirac representation of quantum mechanics these matrices represent γ^5, $\mathrm{i}\gamma^0\gamma^5$, and γ^0, respectively.

In spherical coordinates the sigma matrices are:

$$\sigma_r = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} & 0 & 0 \\ \sin\theta\, e^{i\phi} & -\cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta\, e^{-i\phi} \\ 0 & 0 & \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\sigma_\theta = \begin{pmatrix} -\sin\theta & \cos\theta\, e^{-i\phi} & 0 & 0 \\ \cos\theta\, e^{i\phi} & \sin\theta & 0 & 0 \\ 0 & 0 & -\sin\theta & \cos\theta\, e^{-i\phi} \\ 0 & 0 & \cos\theta\, e^{i\phi} & \sin\theta \end{pmatrix} \tag{275}$$

$$\sigma_\phi = \begin{pmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{pmatrix}$$

The operator $\boldsymbol{\sigma}\cdot\nabla\psi$ therefore yields:

$$\begin{aligned} \boldsymbol{\sigma}\cdot\nabla\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} &= \begin{pmatrix} \cos\theta\left[\partial\psi_1/\partial r\right]+\sin\theta\, e^{-i\phi}\left[\partial\psi_2/\partial r\right] \\ \sin\theta\, e^{i\phi}\left[\partial\psi_1/\partial r\right]-\cos\theta\left[\partial\psi_2/\partial r\right] \\ \cos\theta\left[\partial\psi_3/\partial r\right]+\sin\theta\, e^{-i\phi}\left[\partial\psi_4/\partial r\right] \\ \sin\theta\, e^{i\phi}\left[\partial\psi_3/\partial r\right]-\cos\theta\left[\partial\psi_4/\partial r\right] \end{pmatrix} \\ &+\frac{1}{r}\begin{pmatrix} -\sin\theta\left[\partial\psi_1/\partial\theta\right]+\cos\theta\, e^{-i\phi}\left[\partial\psi_2/\partial\theta\right] \\ \cos\theta\, e^{i\phi}\left[\partial\psi_1/\partial\theta\right]+\sin\theta\left[\partial\psi_2/\partial\theta\right] \\ -\sin\theta\left[\partial\psi_3/\partial\theta\right]+\cos\theta\, e^{-i\phi}\left[\partial\psi_4/\partial\theta\right] \\ \cos\theta\, e^{i\phi}\left[\partial\psi_3/\partial\theta\right]+\sin\theta\left[\partial\psi_4/\partial\theta\right] \end{pmatrix} + \frac{1}{r\sin\theta}\begin{pmatrix} -ie^{-i\phi}\left[\partial\psi_2/\partial\phi\right] \\ ie^{i\phi}\left[\partial\psi_1/\partial\phi\right] \\ -ie^{-i\phi}\left[\partial\psi_4/\partial\phi\right] \\ ie^{i\phi}\left[\partial\psi_3/\partial\phi\right] \end{pmatrix} \end{aligned} \tag{276}$$

In cylindrical coordinates ($r_\perp, \phi, z$) the matrices are:

$$\sigma_{r_\perp} = \sigma_x \cos\phi + \sigma_y \sin\phi = \begin{pmatrix} 0 & e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_\phi = -\sigma_x \sin\phi + \sigma_y \cos\phi = \begin{pmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3.9.3. Wave Properties of Matter

We have shown that classical wave theory can describe Fermion dynamics. This result lends support to recent efforts to revive the classical aether (or ether) as a medium of propagation of matter waves. Duffy (2006) has surveyed modern aether theory.

The model of vacuum as an ideal elastic solid was quite successful in explaining classical properties of light in the 19th century (see e.g. Whittaker (1951)). Quantum effects are only apparent in interactions with matter, which might be interpretable as classical soliton waves. At present there appears to be no satisfactory description of rotational waves in an ideal elastic medium. Kleinert (1989) attempted to include rotations in the elastic energy but was compelled to introduce new elastic constants dependent on an arbitrary scale length. Close (2002) showed that torsion waves (with rotation axis parallel to wave velocity) can be described by a Dirac equation. In this book we use a wave equation with convection terms as the classical basis for the quantum mechanical momentum and spin operators. Schmelzer (2009) recently demonstrated that a cellular lattice model can yield the same group structure as the Standard Model. This model is astonishingly similar to the rotating elastic cell model which Maxwell used to derive the equations of electromagnetism (though the rolling particles bordering Maxwell's cells were replaced by an unspecified material between the lattice cells).

Many physical properties of matter can be derived from a wave model of matter. The Uncertainty Principle applies to all classical waves and represents a basic property of Fourier transformations. Lorentz invariance is also a property of waves, and Special Relativity is therefore a consequence of any wave theory of matter. For example, the relativistic phenomenon of time dilation is simply explained by the fact that stationary soliton waves execute periodic

orbits (e.g. circles) whereas moving solitons execute orbits which have longer wave paths in each cycle (e.g. spiral or cycloidal). Hence a moving clock which counts soliton wave orbits ticks faster than a similar moving clock. Absolute motion with respect to the aether would not be detectable because without prior knowledge of absolute motion it is unknown whether a signal is Doppler shifted at the source or the receiver, or both.

There has been considerable interest in describing elementary particles as soliton (or particle-like) wave solutions of a nonlinear Dirac equation. See Rañada (1983) for a short review. More recent works include Fushchych and Zhdanov (1997), Gu (1998), Bohun and Cooperstock (1999), and Maccari (2006). These efforts all suffer from arbitrariness in the choice of nonlinearity. Identification of the Dirac equation with a second-order classical wave equation provides a simple means for interpreting, literally or analogously, any non-linear terms.

The Klein-Gordon (or relativistic Schrödinger) operator can be factored into a product of two Dirac operators acting on the wave polarization (or amplitude) **a**:

$$\left\{\partial_t^2 - c^2\nabla^2 + M^2\right\}\mathbf{a} = \left\{\gamma_0\partial_t + \gamma_i\partial_i + \mathrm{i}\,M\right\}\left\{\gamma_0\partial_t + \gamma_i\partial_i - \mathrm{i}\,M\right\}\mathbf{a} \tag{277}$$

where the commutation relations are:

$$\begin{aligned} &{\gamma_0}^2 = 1 \\ &\gamma_0\gamma_i = -\gamma_i\gamma_0 \\ &\gamma_i\gamma_j = -\delta_{ij} - \mathrm{i}\,\varepsilon_{ijk}\gamma_k \\ &\mathrm{i}^2 = -1 \end{aligned} \tag{278}$$

The quantities γ_μ and unit imaginary (i) have traditionally been regarded as matrices, but they can also be interpreted geometrically using multivariate vectors [Hestenes 1967, 1973, 1990]. The wave polarization **a** is a classical 3-vector in Galilean space-time. The Minkowski metric of relativity is introduced through the operators.

If we define a wave function:

$$\Psi \equiv \left\{\gamma_0\partial_t + \gamma_i\partial_i - \mathrm{i}\,M\right\}\mathbf{a} \tag{279}$$

then the resultant first-order Dirac equation is equivalent to the original Klein-Gordon equation:

$$\left\{\gamma_0\partial_t + \gamma_i\partial_i + \mathrm{i}\,M\right\}\Psi = 0 \tag{280}$$

In the above case the two Dirac operators have different sign for the mass term. Rowlands [1998, 2005, 2006] and Rowlands and Cullerne [2000] used a combination of multivariate 4-vectors and quaternions to write the Dirac equation in a nilpotent form in which the two successive Dirac operations are identical. This formulation yields an elegant classification of particle states within the Standard Model.

Standard solutions of the Klein-Gordon equation yield different energy eigenvalues than the Dirac equation (see e.g. Schiff [1968]). This result is quite peculiar given the fact that each component of the Dirac wave function actually satisfies the Klein-Gordon equation! Factoring the Klein-Gordon equation cannot change its eigenvalues. The problem is that in the usual analysis of Klein-Gordon, the angular functions are chosen to be eigenfunctions of the squared orbital angular momentum L^2, whereas in the analysis of the Dirac equation the angular functions are eigenvalues of the squared total angular momentum J^2. The difference is not in the equations, but in the choice of angular eigenfunctions. The usual analysis of the Klein-Gordon equation neglects the spin contribution from rotation of wave velocity. These solutions represent bosons with zero spin. Solutions obtained by using angular eigenvalues obtained from Dirac theory represent fermions with spin one-half.

In the next chapter we shall see that a scalar gravitational field and its effect on the space-time metric may be interpreted as a spatially varying light speed. See Whittaker (1954) for the historical development of this idea which originates with Einstein (1911, 1912) and has also been investigated more recently (de Felice (1971), Evans et al (2001)). This interpretation is consistent with general relativity, which also predicts a variation of light speed proportional to the gravitational potential (Einstein 1956). In an elastic solid aether, compression or variations in elasticity imply variable wave speed and hence provide a reasonable physical model for basic gravitational effects.

3.10. Summary

Even if you are a minority of one, the truth is the truth.

— Mohandas Gandhi

In this chapter we interpret the Dirac equation as a classical second-order wave equation for rotational waves in an elastic medium. The first order spatial and temporal derivatives are represented by a bispinor wave function. Half-integer spin is attributable to the co-existence of waves traveling in opposite directions along the gradient axis. The wave function can be factored into constant matrix, a single amplitude, a three-dimensional Lorentz velocity boost, rotation, and an arbitrary change of representation. Wave interference yields both the Pauli exclusion principle and the Lorenz force. The electromagnetic potentials represent wave interference. Interpreting the classical bispinor equation as describing an electron, it is found that the mass is associated with radially inward acceleration of the wave, suggestive of a soliton. The classical theory is consistent with parity conservation. Hence it appears that classical wave theory constitutes an intelligible basis for the physical attributes of matter.

3.11. Suggested Exercises:

1. Verify the correspondence between bispinor and scalar equations in Table I.
2. A Hermitian matrix has the property $\alpha^\dagger = \alpha$. Prove that $\psi^\dagger \alpha \psi$ is real for any Hermitian matrix α.

3. Use Taylor series expansion to verify that a spin 1/2 rotation operator can be written as an exponential: $\exp\{-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}/2\}=\left[\cos(\varphi/2)-\mathrm{i}\hat{\boldsymbol{\varphi}}\cdot\boldsymbol{\sigma}\sin(\varphi/2)\right]$
4. Derive the velocity eigenfunctions ψ_a which satisfy the equation $c\beta_1\boldsymbol{\sigma}\psi_\alpha=\mathbf{v}\psi_\alpha$.
5. Transform the spin matrices into spherical coordinates.
6. Verify the general formula for relativistic addition of velocities.
7. Show that the angular momentum is constant (commutes with Hamiltonian):

$$\frac{d}{dt}\left[\psi^\dagger[\mathbf{L}+\mathbf{S}]\psi\right]=\frac{d}{dt}\left[\psi^\dagger\left[-\mathbf{r}\times\mathrm{i}\nabla+\frac{1}{2}\boldsymbol{\sigma}\right]\psi\right]=0$$

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