Introduction to wave mechanics: Interactions

Robert A Close
Clark College, 1933 Fort Vancouver Way, Vancouver, WA 98663 USA
E-mail: robert.close@classicalmatter.org
(Dated: November 21, 2024)

The first-order quantum mechanical Dirac equation is interpreted as a representation of a second-order vector wave equation for spin angular momentum density (or spin density), the classical vector field whose curl is equal to twice the momentum density. This interpretation describes interactions between particle-like waves, offering students a conceptual bridge between classical physics and quantum mechanics. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials. Classical interpretations of magnetic flux quantization and the Coulomb potential are presented. A classical version of the single-electron Lagrangian for quantum electrodynamics is also presented.

Keywords: classical interpretation, Dirac equation, elastic solid, magnetic flux quantization, quantum electrodynamics, quantum mechanics pedagogy, spin angular momentum, spin density, teaching quantum mechanics, wave mechanics

1. INTRODUCTION

Previous work has proposed that physics students should be introduced to quantum mechanics using the Dirac equation rather than the Schrödinger equation. Unlike the Schrödinger equation, the Dirac equation is compatible with special relativity, and also has a simple classical physics interpretation as a description of spin density ($\mathbf{s}$), the field whose curl is equal to twice the momentum density of a continuous medium. [1–3] The relationship to Dirac wave functions is:

$$\mathbf{s} = \partial_t \mathbf{Q} \equiv \frac{1}{2}\left[\psi^\dagger \boldsymbol{\sigma} \psi\right]; \tag{1a}$$

$$c\nabla \cdot \mathbf{Q} \equiv -\frac{1}{2}\left[\psi^\dagger \gamma^5 \psi\right]; \tag{1b}$$

$$c^2\left\{\nabla \times \nabla \times \mathbf{Q}\right\} \equiv -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right] \times \gamma^5 \boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma} \times \nabla\psi\right\}; \tag{1c}$$

$$0 = \frac{\mathrm{i}c}{2}\nabla \cdot \left\{\left[\nabla\psi^\dagger\right] \times \gamma^5 \boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma} \times \nabla\psi\right\}\ . \tag{1d}$$

Although the use of spinors is beyond typical undergraduate curricula, the above relationships between vectors and spinors is easily derived as a three-dimensional generalization of the one-dimensional wave equation. [1–3] The Dirac formalism, in addition to quantum mechanical application, has a variety of applications in describing classical physics. [1–11] There has also been observation of quantum mechanical behavior of classical systems. Quantum statistics have been observed in experiments using silicone droplets bouncing on a vibrating water tank. [12–18] Such experiments provide a physical realization of Bohmian mechanics, or pilot-wave theory. [19–21]

This work examines how classical wave interactions give rise to the Pauli exclusion principle and electromagnetic potentials. We discuss magnetic flux quantization and magnetic flux and electric charge of an electron. Finally, we relate the classical Lagrangian to that of quantum electrodynamics.

2. WAVE INTERACTIONS

Suppose we have two Dirac wave functions ψ_A and ψ_B, representing particle-like waves A and B. Adding the wave functions yields a total wave function ψ_T satisfying:

$$\begin{aligned}\psi_T^\dagger\boldsymbol{\sigma}\psi_T &= (\psi_A+\psi_B)^\dagger\boldsymbol{\sigma}(\psi_A+\psi_B)\\ &= \psi_A^\dagger\boldsymbol{\sigma}\psi_A + \psi_B^\dagger\boldsymbol{\sigma}\psi_B + \psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A\,.\end{aligned} \tag{2}$$

Since the spins must be additive, the total wave function is not generally the sum of the individual wave functions. However, we can treat the wave functions as being independent if the interference terms cancel [1]. This cancelation imposes a vector constraint on the wave functions:

$$\psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A = 0\,. \tag{3}$$

Assuming either of the waves to be a spin eigenfunction everywhere, one component of this constraint requires the wave functions to anti-commute:

$$\psi_A^\dagger\psi_B + \psi_B^\dagger\psi_A = 0\,. \tag{4}$$

For waves representing identical particles, this is the Pauli exclusion principle. Hence we can conclude that standing waves described by spin eigenfunctions are fermions.

Even if neither particle is a spin eigenfunction, we can construct a modified particle $\psi'_B = \sigma_B\psi_B$ where $\sigma_B(\mathbf{r},t) = \hat{\mathbf{s}}_B\cdot\boldsymbol{\sigma}$ is the spin matrix aligned with the spin direction $\hat{\mathbf{s}}_B(\mathbf{r},t)$ of particle B. The wave function ψ'_B yields the same spin everywhere as ψ_B, and thus is physically equivalent. Using ψ'_B in Eq. 3 again yields Eq. 4 for the component aligned with the spin of particle B.

The anti-commutation of wave functions is not true in general, but we can force the cancellation by introducing a phase shift at each point between the two wave functions. Such phase shifts have no effect on the actual dynamics of the total wave, but allow us to pretend that each particle wave maintains its separate identity even though there is actually only one combined wave. Of course, this procedure is only valid if the particles interact weakly enough

to remain distinguishable during the interaction. This limitation does not invalidate the basic premise that physical quantities are fully determined by the spin density field.

The phase shift (δ) is determined from the constraint:

$$\mathrm{Re}(\psi_A^\dagger \exp(\mathrm{i}\delta)\psi_B) = 0, \tag{5}$$

or

$$\mathrm{Re}(\psi_A^\dagger \psi_B)\cos\delta - \mathrm{Im}(\psi_A^\dagger \psi_B)\sin\delta = 0\,. \tag{6}$$

This yields:

$$\tan\delta = \frac{\mathrm{Re}(\psi_A^\dagger \psi_B)}{\mathrm{Im}(\psi_A^\dagger \psi_B)}\,. \tag{7}$$

If we define $\exp(\mathrm{i}\beta) = \psi_A^\dagger \psi_B / |\psi_A^\dagger \psi_B|$, then:

$$\tan\delta = \cot\beta\,. \tag{8}$$

and:

$$\exp(\mathrm{i}\delta) = \pm\mathrm{i}\exp(-\mathrm{i}\beta) = \mp\mathrm{i}\,\frac{\psi_A^\dagger \psi_B}{|\psi_A^\dagger \psi_B|}\,. \tag{9}$$

Therefore the phase angles are related by:

$$\delta = \frac{\pi}{2} - \beta \pm n\pi \tag{10}$$

where n is an integer. Note that δ is only unique within an arbitrary integer multiple of π.

Suppose we start with two wave functions ψ'_A and ψ'_B, initially non-overlapping and normalized to one ($\hbar$ will be multiplied explicitly to provide the dimension of angular momentum). We will assume that each particle is shifted by a phase attributable to the other particle: $\psi_A = \psi'_A \exp(\mathrm{i}\delta_A)$ and $\psi_B = \psi'_B \exp(\mathrm{i}\delta_B)$, where the primed variables have zero interference ($\delta_A - \delta_B = \delta$). As they approach each other, the total wave function is $\psi_T = \psi_A + \psi_B$. The phase shifts satisfy:

$$\exp(\mathrm{i}(\delta_A - \delta_B)) = \mathrm{i}\,\frac{\psi_B^\dagger \psi_A}{|\psi_A^\dagger \psi_B|} \equiv \exp(\mathrm{i}(\varphi_B - \varphi_A + \pi/2)) \tag{11}$$

where φ_A and φ_B are phases associated with particles A and B, respectively. This construction assumes that variations of phase of each component of a particle-like wave are attributable to a common phase factor. The phase shifts must be weighted so that the weaker wave is more affected by the stronger wave than vice-versa. A simple choice would be to weight the phase shifts by the relative magnitudes of each wave:

$$\begin{aligned}
\delta_A &= \frac{|\psi_B|^2}{|\psi_A|^2 + |\psi_B|^2}(\varphi_B - \varphi_A + \pi/2)) \\
\delta_B &= -\frac{|\psi_A|^2}{|\psi_A|^2 + |\psi_B|^2}(\varphi_B - \varphi_A + \pi/2))
\end{aligned} \tag{12}$$

Since the $\pi/2$ phase shift is constant, we will omit it from the following analysis of derivatives of the phase shifts.

The original wave functions satisfy the free-particle wave equation, e.g.

$$\mathrm{i}\hbar\partial_t(\exp[-\mathrm{i}\delta_A]\psi_A) = H_0 \exp[-\mathrm{i}\delta_A]\psi_A. \tag{13}$$

We take ψ'_A to be an electron wave function with free particle hamiltonian $H_0\psi'_A = (-c\gamma^5\boldsymbol{\sigma}\cdot\mathrm{i}\hbar\nabla + m_e c^2\gamma^0)\psi'_A$.

Expanding the Dirac equation for ψ_A yields:

$$\left(\hbar(\mathrm{i}\partial_t + [\partial_t\delta_A]) = \hbar c\gamma^5\boldsymbol{\sigma}\cdot(-\mathrm{i}\nabla - [\nabla\delta_A]) + m_e c^2\gamma^0\right)\psi_A\,, \tag{14}$$

where square brackets indicate that the derivatives apply only to the variables inside the brackets.

The modified Hamiltonian is:

$$H\psi_A = \mathrm{i}\hbar\partial_t\psi_A = -\hbar[\partial_t\delta_A]\psi_A + c\gamma^5\boldsymbol{\sigma}\cdot\hbar(-\mathrm{i}\nabla - [\nabla\delta_A])\psi_A + m_e c^2\gamma^0\psi_A\,. \tag{15}$$

The modified momentum density is:

$$\psi_A^\dagger \exp\left[\mathrm{i}\delta_A\right](-\mathrm{i}\hbar\nabla)\exp\left[-\mathrm{i}\delta_A\right]\psi_A = \psi_A^\dagger(-\mathrm{i}\hbar\nabla - \hbar\nabla\delta_A)\psi_A \tag{16}$$

The wave force density, or time derivative of momentum density, is found from the partial time derivative and the commutator of the hamiltonian with the momentum operator:

$$\begin{aligned} d_t P_i &= \psi_A^\dagger(-\hbar\partial_t\partial_i\delta_A + \frac{\mathrm{i}}{\hbar}[H, (-\mathrm{i}\hbar\partial_i - \hbar\partial_i\delta_A)])\psi_A \\ &= \psi_A^\dagger\left[-\hbar\partial_t\partial_i\delta_A + \hbar\partial_i(\partial_t\delta_A) - \hbar c(\gamma^5\sigma_j\partial_j)\partial_i\delta_A + \hbar c\partial_i(\gamma^5\sigma_j\partial_j\delta_A)\right]\psi_A \\ &= \psi_A^\dagger\left[\hbar(\partial_i\partial_t - \partial_t\partial_i)\delta_A + \hbar c\gamma^5\sigma_j(\partial_i\partial_j - \partial_j\partial_i)\delta_A)\right]\psi_A \end{aligned} \tag{17}$$

Note that there is no change in momentum density if the derivatives commute everywhere. Hence plane waves would not be affected by the presence of other plane waves. However, the derivatives do not generally commute. The spatial derivatives do not generally commute because the phases are multivalued. And the time derivative is not independent of the spatial derivatives because each particle's phase pattern moves with the particle. For example, a wave pulse $f(x,t) = f_0\exp\left[-(x-ct)^2\right]$ moves along the x-axis with speed c. The pulse is defined by its shape $f(x')$, but the variable $x' = (x-ct)$ is not independent of time. In this case $\partial_t f = -c\partial_{x'}f$. Likewise, a particle with phase factor $f = \cos(m\phi' - \omega t)$, with ϕ' defined relative to the center of the rotating wave pattern, is also defined by trajectories of constant phase. In this case $\partial_t f = -(\omega/m)\partial_{\phi'}f$. Keep in mind that our attribution of phase shifts must be consistent with our artificial decomposition of the wave into "particles".

To simplify analysis, we define the vector potential by $q\mathbf{A} \equiv \hbar\nabla\delta_A$, the electric potential by $q_A\Phi = -\partial_t\delta_A$, the charge density by $\rho_A \equiv q_A\psi_A^\dagger\psi_A$, and the current density by $\mathbf{J}_A \equiv \psi_A^\dagger q_A c\gamma^5\boldsymbol{\sigma}\psi_A$. The rate of change of momentum density is:

$$d_t P_i = \rho_A(-\partial_i\Phi_A - \partial_t A_i) + J_j(\partial_i A_j - \partial_j A_i) \tag{18}$$

In vector form:

$$d_t\mathbf{P} = \rho_A(-\nabla\Phi - \partial_t\mathbf{A}) + \mathbf{J}_A\times(\nabla\times\mathbf{A})\,. \tag{19}$$

This is equivalent to the Lorentz force with:

$$\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}\,, \tag{20a}$$

$$\mathbf{B} = \nabla\times\mathbf{A}\,. \tag{20b}$$

Others have similarly identified the vector potential $\mathbf{A}$ as the gradient of a multivalued field.[22–24] The curl of such gradients need not be identically zero. This interpretation is also consistent with Synge's "primitive quantization" in which Planck's constant h represents the action for a single wave cycle. [25]

Note that since forces arise only from non-commuting variables, the electric force on particle A is proportional to $|\psi_A|^2|\psi_B|^2/(|\psi_A|^2 + |\psi_B|^2)$, and is everywhere equal and opposite to the electric force on particle B. The magnetic forces on A and B are not necessarily locally equal and opposite, as is also the case in classical physics.

2.1. Charge and Parity

Thus far we have offered no explanation for positive and negative charge. However, it is well known that vector spherical harmonics have odd parity for odd integer angular quantum numbers, and even parity for even integer angular quantum numbers. Hence vector spherical harmonics have distinct mirror images only for odd integer angular quantum numbers. Since the bispinor wave functions transform under rotations with half the phase change of the vector wave functions, we therefore expect that elementary bispinors with half-integer quantum numbers have distinct mirror images, whereas elementary bispinors with whole integer quantum numbers do not have distinct mirror images. This is consistent with the fact that all elementary fermions have distinct antiparticles, and nearly all elementary bosons do not have distinct antiparticles (the W^+ and W^- are each other's antiparticle, but perhaps they are not described by simple spherical harmonics).

The conventional transformation between matter and antimatter includes spatial inversion but is not equivalent to it. However, the conventional parity operator was derived assuming the underlying physical space to be a Minkowski spacetime, whereas the analysis here starts from a Galilean spacetime. Hence for our purposes we simply assume that matter and antimatter, and hence positive and negative charges, are simply mirror images of each other.

2.2. Magnetic Flux Quantization

Suppose wave B is a macroscopic wave ($|\psi_B|^2 >> |\psi_A|^2$) with azimuthal dependence $\psi_B \sim \exp(\mathrm{i}(m_B\phi_B - \omega_B t)$, where ϕ_B is the azimuthal angle around a local axis z_B and m_B is a half or whole integer azimuthal angular quantum number. The angular quantum number and frequency ω_B are doubled for the vector wave function. The phase shift of ψ_A would then vary by $m_B\pi$ along a closed path around the z_B-axis:

$$\oint \nabla\delta_A \cdot d\boldsymbol{\ell} = m_B\pi \tag{21}$$

Stoke's law then yields quantization of magnetic flux:

$$\iint \mathbf{B}\cdot\hat{\mathbf{n}}\, dS = \oint \mathbf{A}\cdot d\ell = m_B\pi\frac{\hbar}{e} = m_B\frac{h}{e}\,. \tag{22}$$

Allowing m_B to be half-integer, this is the same value as found in superconductors. For superconductors, the phase comes from the electron pair wave function with $m_B = 1$ rather than from the macroscopic wave function attributable to all the other particles, and the charge is doubled. (((needs reference)))

This classical quantization of magnetic flux is consistent with de Broglie's observation in a 1963 interview that "... in quantum phenomena one obtains quantum numbers, which are rarely found in mechanics but occur very frequently in wave phenomena and in all problems dealing with wave motion." [26]

2.3. Electron Interactions

Alternatively, suppose A and B are electrons with phase factor $\exp(\mathrm{i}(m_B\phi_B - \omega_B t))$ for ψ_B and similarly for ψ_A. The angular frequency of the bispinor wave function is related to mass by $\hbar\omega_B = m_e c^2$. The angular frequency of the vector wave function would be twice this (and angular momentum would be $\hbar/2$).

For simplicity, we define $r_B = |\mathbf{r} - \mathbf{r}_B|$ to be the distance from the center of particle B at $\mathbf{r}_B$. The distance r_A is similarly defined for particle A. The distance between the centers of the two particles is r_{AB}.

The phase shift of ψ_A is:

$$\delta_A = \frac{|\psi_B|^2}{|\psi_A|^2 + |\psi_B|^2}[(m_B\phi_B - \omega_B t)) - (m_A\phi_A - \omega_A t))] \tag{23}$$

2.3.1. Magnetic Flux

The computed magnetic flux around B will vary with distance from the particle center. We are interested in the phase shift of A attributable to particle B, near the center of A. Near $r_A = 0$ we have $|\psi_A|^2 >> |\psi_B|^2$. Hence we can approximate the phase shift as:

$$\delta_A \approx \left(\left[\frac{|\psi_B|^2}{|\psi_A|^2}\right]_{\mathbf{r}\approx\mathbf{r}_A}\right)(m_B\phi_B - \omega_B t) \tag{24}$$

Assume that each charge density has a peak value ρ_M and asymptotic radial decay of $\rho_M\alpha/(\kappa r)$.

$$\delta_A \approx \left(\frac{\alpha}{\kappa r_B}\right)(m_B\phi_B - \omega_B t) \tag{25}$$

This is the phase shift in the vicinity of $r_B \approx r_{AB}$ where $|\psi_A|^2 >> |\psi_B|^2$. The magnetic vector potential is:

$$\frac{e}{\hbar}\mathbf{A} = \left(\frac{\alpha}{\kappa r_B}\right)\left(\frac{m_B}{r_B\sin(\theta_B)}\right)\hat{\boldsymbol{\phi}}_B - \frac{\alpha}{\kappa r_B^2}(m_B\phi_B - \omega_B t)\hat{\mathbf{r}} \tag{26}$$

If we had kept the contribution of particle A in the phase shift, there would also be a term proportional to $\hat{\boldsymbol{\phi}}_A$, which would average to nearly zero when integrating near $r_A = 0$.

Magnetic flux is due to the azimuthal component of the vector potential, so we ignore the radial term:

$$\frac{e}{\hbar}\mathbf{A} = \left(\frac{\alpha}{\kappa r_B}\right)\left(\frac{m_B}{r_B\sin(\theta_B)}\right)\hat{\boldsymbol{\phi}}_B \tag{27}$$

Given the expression for $\mathbf{A}$, the effective magnetic field at finite distance r_B from the center of B is:

$$\mathbf{B} = \frac{\hbar\,\alpha m_B}{e\kappa r_B^3 \sin(\theta_B)}\hat{\boldsymbol{\theta}}_B \tag{28}$$

This is not a simple dipole field.

For $m_B = 1/2$, the magnetic flux through the plane $\sin(\theta_B) = 1$ inside radius r_B is:

$$\oint (\mathbf{A}\cdot d\ell) = -\frac{\hbar\alpha}{e\kappa r_B}\pi \tag{29}$$

Taking $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ to the be fine structure constant and $\kappa = m_e c/\hbar$, the radial variation of this flux is the same as for an electron. Of course, the radial dependence was chosen to give this result. Next, we will show that this flux is consistent with the charge of the electron.

2.3.2. Electric Field

To calculate an electric force, we note that the phase velocity for the phase shift δ_A is:

$$\mathbf{v} = \frac{\omega_B}{m_B} r\sin\theta_B \hat{\boldsymbol{\phi}}_B \tag{30}$$

For simplicity, we neglect any motion of particle B. Changes of phase are attributable to wave propagation. Using $\partial_t\delta_A = -\mathbf{v}\cdot\nabla\delta_A$ yields:

$$eE_i = \hbar((\partial_i v_j)\partial_j + v_j(\partial_i\partial_j - \partial_j\partial_i))\delta_A \tag{31}$$

The second term cannot contribute to the field because the phase velocity is zero at the point where the derivatives do not commute. The radial component is thus (with $\phi = \phi_B$ defined relative to the center of ψ_B):

$$eE_r = \hbar(\partial_r v_\phi)A_\phi \tag{32}$$

Direct substitution yields:

$$eE_r = \frac{\hbar\omega_B\,\alpha}{\kappa r_B^2} \tag{33}$$

Substituting $\kappa = \omega_B/c$ yields:

$$\mathbf{E} = \frac{\hbar c\,\alpha}{e r_B^2}\hat{\mathbf{r}}_B = \frac{e}{4\pi\epsilon_0 r_B^2}\hat{\mathbf{r}}_B \tag{34}$$

This is the usual electric field of a point charge located at $r_B = 0$. Note that this result is independent of the azimuthal quantum number and the frequency (or mass).

Recall that we neglected any contribution from the phase of particle A. Since the electric field is radial, any contribution from the phase shift proportional to ϕ_A would not affect the bulk motion of particle A.

A similar relationship between electric charge and magnetic flux was obtained by Jehle, who assumed a spatially diffuse magnetic flux quantum rotating with angular frequency $\omega = 2m_e c^2/\hbar$. [22, 23] In that work, the sign of the charge depends on the relative orientation of the magnetic flux and the direction of rotation. In the work presented here, this corresponds to the relative sign of the spatial and temporal phases: $(m_B\phi \pm \omega t)$.

2.4. Maxwell's Equations

The electromagnetic fields defined above are also subject to Maxwell's equations. The definitions of $\mathbf{E}$ and $\mathbf{B}$ imply Faraday's Law and Gauss' magnetic law:

$$\nabla\times\mathbf{E} = -\partial_t\mathbf{B}\,, \tag{35a}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{35b}$$

Gauss' electric law and Ampere's law define the charge and current densities (ρ_e and $\mathbf{J}$, respectively):

$$\nabla \cdot \mathbf{E} = -(\nabla \cdot \partial_t \mathbf{A} + \nabla^2 \Phi) \equiv \frac{\rho_e}{\epsilon_0}\,, \tag{36a}$$

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} &= \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}(\partial_t^2 \mathbf{A} + \partial_t \nabla \Phi) \\ &\equiv \mu_0 \mathbf{J}\,. \end{aligned} \tag{36b}$$

The last equality is equivalent to the usual expression:

$$\frac{1}{c^2}\partial_t^2 \mathbf{A} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J} \tag{37}$$

subject to the Lorentz condition:

$$\frac{1}{c}\partial_t \Phi + \nabla \cdot \mathbf{A} = 0 \tag{38}$$

As derived above, the electric field of a spherical harmonic wave corresponds to that of a point charge. These definitions of charge and current densities are consistent with the continuity equation:

$$\partial_t \rho_e + \nabla \cdot \mathbf{J} = 0\,. \tag{39}$$

There is a discrepancy between the point-like source charge density of Maxwell's equations, and the object charge density of wave mechanics that we assumed earlier. In wave mechanics, the object charge density is "smeared out" throughout the wave function. It is similarly smeared out in quantum mechanics, in which case the smearing is attributed to uncertainty of position.

Hence particle-like waves in an elastic solid can behave like fermions, with electromagnetic potentials derived from phase shifts that result from wave interference.

2.5. Quantum Electrodynamics

It is customary in quantum mechanics textbooks to define $\bar{\psi} \equiv \psi^\dagger \gamma^0$, replace $\psi^\dagger$ with $\bar{\psi}\gamma^0$, and define the "4-vector" of matrices $\gamma^\mu \equiv (\gamma^0, \gamma^0\gamma^5\boldsymbol{\sigma})$. The 4-potential is $A_\mu = (\Phi, -\mathbf{A})$ and the 4-current $(\rho, \mathbf{J})$ is $J^\mu = q\bar{\psi}\gamma^\mu\psi$. These changes of variables are intended to make the theory look more "relativistic". It is also common to use "natural" units with $\mu_0 = \epsilon_0 = c = 1$. Using this notation with $\partial_\mu = (\partial_t, \nabla)$, the Lagrangian density for two interacting electrons is:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \bar{\psi}_B[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_B]\psi_B\,. \tag{40}$$

Separating the interaction of particle B yields:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \bar{\psi}_B[\gamma^\mu(\mathrm{i}\partial_\mu) - m_B]\psi_B - J^\mu A_\mu\,. \tag{41}$$

Since the Dirac equation is satisfied for each particle, this is equivalent to:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + J^\mu A_\mu - J^\mu A_\mu\,. \tag{42}$$

Relationships between potentials and sources are given in Eqs. 36. Assuming time-independence with zero divergence of the vector potential and zero curl of the electric field, the sources become:

$$\frac{\rho_e}{\epsilon_0} = -\nabla^2 \Phi\,, \tag{43a}$$

$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A})\,. \tag{43b}$$

Therefore:

$$J^\mu A_\mu = -\Phi \nabla^2 \Phi - \mathbf{A} \cdot (\nabla \times \nabla \times \mathbf{A})\,. \tag{44}$$

According to Green's first identity:

$$-\int_V \Phi \nabla^2 \Phi dV = \int (\nabla\Phi)^2 dV - \int_{\partial V} \Phi \mathbf{n} \cdot \nabla \Phi dS\,. \tag{45}$$

Similarly:

$$-\int_V \mathbf{A}\cdot(\nabla\times\nabla\times\mathbf{A})dV = -\int_V (\nabla\times\mathbf{A})^2 dV + \int_{\partial V}\mathbf{A}\times(\nabla\times\mathbf{A})dS\,. \tag{46}$$

Using the definitions of **E** and **B** while neglecting boundary integrals yields:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + (E^2 - B^2) - J^\mu A_\mu \tag{47}$$

This differs from the (non-quantized) Lagrangian density of quantum electrodynamics (QED) by a factor of 1/2 in front of $(E^2 - B^2)$. This difference is resolved by the fact that when varying the potentials A_μ, the source densities J^μ should be regarded as functions of A_μ. However, it is conventional to vary the potentials independently of the source densities, yielding only half of the correct value. When computing variations of E^2 and B^2, both factors in E^2 (and B^2) are varied. To eliminate this double-counting and be consistent with independent variation of the potentials, a factor of 1/2 must be introduced:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \frac{1}{2}(E^2 - B^2) - J^\mu A_\mu \tag{48}$$

This is the Lagrangian density of non-quantized QED, in which a single charged fermion interacts with an electromagnetic field. Generalization to multiple interacting particles requires a quantization procedure with raising and lowering operators to allow for changes in the numbers of particles.

3. DISCUSSION

We have outlined a similarity between a classical model of interacting waves in an elastic solid to quantum electrodynamics (QED). This interpretation of QED, and by extension the Standard Model, is that it represents a decomposition of the classical spin density field into interacting elementary particles. Others have also associated quantum mechanical behavior with waves in an elastic solid. [10, 27–30]

4. CONCLUSIONS

This paper describes interactions of classical waves of spin density. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials, with suggested interpretations of magnetic flux quantization and the Coulomb potential. The Lagrangian density of single-fermion quantum electrodynamics is also given a classical physics interpretation. Hence classical wave theory offers insight into the physical basis for relativistic quantum mechanics.

[1] R. A. Close, in *Ether Space-time and Cosmology*, Vol. 3, edited by M. C. Duffy and J. Levy (Apeiron, Montreal, 2009) pp. 49–73.
[2] R. A. Close, Adv. Appl. Clifford Al. **21**, 273 (2011).
[3] R. A. Close, Elect. J. Theor. Phys. **12**, 43 (2015).
[4] S. Matsutani, J. Phys. Soc. Jpn. **61**, 3825 (1992).
[5] S. Matsutani and H. Tsuru, Phys. Rev. A **46**, 11441147 (1992).
[6] S. Matsutani, Phys. Lett. A **189**, 27 (1994).
[7] R. A. Close, Found. Phys. Lett. **15**, 71 (2002).
[8] A. G. Kyriakos, Apeiron **11**, 330 (2004).
[9] M. Arminjon, FPL **19**, 225 (2006).
[10] P. A. Deymier, K. Runge, N. Swinteck, and K. Muralidharan, J. Appl. Phys. **115**, 163510 (2014).
[11] M. Yousefian and M. Farhoudi, "QED treatment of linear elastic waves in asymmetric environments," arXiv:1912.03272v4 [physics.class-ph] (24 May 2020).
[12] Y. Couder and E. Fort, Phys. Rev. Lett. **97**, 41541010 (2006).
[13] D. M. Harris, J. Moukhtar, E. Fort, Y. Couder, and J. W. M. Bush, Phys. Rev. E **88**, 011001 (2013).
[14] A. Eddi, E. Fort, F. Moisy, and Y. Couder, Phys. Rev. Lett. **102**, 240401 (2009).
[15] E. Fort, A. Eddi, A. Boudaoud, J. Moukhtar, and Y. Couder, Proc. Natl Acad. Sci. **107**, 1751517520 (2010).
[16] A. Eddi, J. Moukhtar, S. Perrard, E. Fort, and Y. Couder, Phys. Rev. Lett. **108**, 264503 (2012).

[17] R. Brady and R. Anderson, "Why bouncing droplets are a pretty good model of quantum mechanics," arXiv:1401.4356 [quant-ph] (16 Jan 2014).
[18] J. W. M. Bush, Annu. Rev. Fluid Mech. **47**, 269 (2015).
[19] E. Madelung, Z. Phys. **40**, 322 (1927).
[20] L. de Broglie, in *Electronset photons: rapports et discussions du cinqui'eme conseil dephysique* (Gautier-Villars, Paris, 1928).
[21] D. A. Bohm, Phys. Rev. **85**, 166, 180 (1952).
[22] H. Jehle, Phys. Rev. D **3**, 306 (1971).
[23] H. Jehle, Phys. Rev. D **6**, 441 (1972).
[24] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (World Scientific, Singapore, 2008) p. 119.
[25] J. L. Synge, *Geometrical Mechanics and De Broglie Waves* (Cambridge University Press, Cambridge, 1954) p. 113.
[26] J. J. O'Connor and E. F. Robertson, "Louis victor pierre raymond duc de broglie," https://mathshistory.st-andrews.ac.uk/Biographies/Broglie/ (May 2001).
[27] M. Danielewski, Z. Naturforsch **62**, 564 (2007).
[28] M. Danielewski and L. Sapa, Bulletin of Cherkasy University **1**, 22 (2017).
[29] I. Schmeltzer, Found. Phys. **39**, 73 (2009).
[30] I. Schmeltzer, in *Horizons in World Physics*, Vol. 278, edited by A. Reimer (Nova Science Publishers, 2012).