The Different Meanings of “Spin”

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The word “spin” has different meanings in different contexts. This article explains three different meanings of the word “spin”. The first is kinetic spin, which refers to angular velocity. The second is geometric spin, which is related to the pointwise symmetry of mathematical basis states under rotation. Sample basis states are described for integer and half-integer rotational quantum numbers. The third type of spin is called dynamic spin, which is the intrinsic or conjugate angular momentum associated with rotational motion of an inertial substance. These different meanings are explained with mathematical descriptions of physical processes.

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I. INTRODUCTION

The term “spin” is used in a variety of contexts in physics and mathematics, so it is important to know what the word means in each context. There are at least three different meanings: (1) kinetic spin, which is related to rotation rate or angular velocity, (2) geometric spin, which is related to transformation under rotations, and (3) dynamic spin, which is related to angular momentum and torque. This paper will explain these three meanings with examples for each.

II. KINETIC SPIN

The colloquial use of the word “spin” refers to rotation of an object or fluid. We say that a top or pinwheel “spins”. This means that it has a non-zero angular velocity. In fluid mechanics, “spin” sometimes refers to the angular velocity itself ($\mathbf{w}$), which is half the curl of the velocity field ($\mathbf{v}$):

$$\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v} \tag{1}$$

Vorticity ($\mathbf{W}$) is equal to twice the angular velocity.

Alternatively, using the symbol ∂_l to denote the partial derivative with respect to coordinate x^l, the “spin tensor” can be defined as:[1]

$$w_{kl} = \frac{1}{2}(\partial_l v_k - \partial_k v_l) \tag{2}$$

with vorticity equal to $W^k = 2w^k = \epsilon^{klm} w_{ml}$.

The term “spin” as used is politics is related to the physical notion of rotation. It refers to interpretation of events from a different viewpoint or “angle”.

III. GEOMETRIC SPIN

In mathematics, the ”spin” of a set of basis states refers to the effects of rotations on the basis states. This concept of spin is closely related to the order of rotational symmetry of planar geometric shapes. This is the (integer) number of times a shape repeats during rotation by 2π radians. Like the geometric order of rotational symmetry, the geometric spin is quantized. However, the spin quantum numbers may be odd half-integers as well as integers.

A spin-1 system utilizes independent states (vectors) separated by $\pi/2$ radians and symmetric about some axis only after rotation of 2π radians. A spin-1 state repeats only once after a complete 2π-radian rotation. Geometrically, this corresponds to a directed line segment, which is a common symbol for a vector. A spin-2 system utilizes independent states separated by $\pi/4$ radians and symmetric only after rotation of π radians. Geometrically, this corresponds to a directionless line segment. And so on for higher integers, with spin-3 corresponding to an equilateral triangle, spin-4 corresponding to a square, etc.

A spin-0 system utilizes scalar functions that are independent of rotation. Geometrically, it corresponds to a point. Although one could argue that a point repeats its shape infinitely many times during a 2π rotation, the interpretation of the order of rotational symmetry is that since it never changes its shape, it also doesn’t ”repeat” its shape.

A spin-1/2 system consists of independent states (spinors or bispinors) that have independent states separated by π radians and are symmetric (i.e. identical) about some axis only after rotation of 4π radians. Although a mobius strip might seem like a geometric analogy, the strip actually returns to its original state after 2π rotation (which is not the same as tracing a line along the surface). A spin-3/2 system utilizes independent states separated by $\pi/3$ radians and symmetric about some axis only after rotation of $4\pi/3$ radians.

One possible definition of geometric spin “s” consistent with the spin values described above is:

$$s = \frac{2\pi}{\text{Angle of symmetry}} \tag{3}$$

Note that rotations above are defined “about some axis”. There may be other axes for which the symmetry angle is larger (perhaps infinite, indicating that rotation about that axis does not change the state). To find the spin, find the axis with the smallest angle of symmetry. Rotations about other axes correspond to combinations of different values of ”m_s”, which can vary in integer steps with $-s \leq m_s \leq s$. The special case of $m_s = 0$ corresponds to rotation about an axis of azimuthal symmetry.

In this paper, we will mostly consider the two-dimensional plane of maximal variation with rotation. This corresponds to $m_s = \pm s$.

The "state" of a system is a mathematical expression for some physical variable(s). We will denote an arbitrary state with spin s by the symbol Ξ_s. Any state can be expressed as a linear superposition of "basis" states. Suppose we start with a pure basis state $\mathbf{A}$ with amplitude a_0, with any other basis states having amplitude of zero. The amplitude of $\mathbf{A}$ varies under rotation by angle δ as $a_0 \cos{(m_s\delta)}$. The initial basis state has phase $m_s\delta = 0$ and $\cos{(m_s\delta)} = 1$. When the phase is $m_s\delta = 2\pi$, we again have $\cos{(m_s\delta)} = 1$ and the amplitude of $\mathbf{A}$ is again equal to a_0. When the phase is $m_s\delta = \pi/2$, the amplitude of $\mathbf{A}$ is zero. This implies that we have reached a new basis state: call it $\mathbf{B}$ with amplitude b. In two dimensions, we have:

$$\Xi = a_0 \cos{(m_s\delta)}\mathbf{A} + a_0 \sin{(m_s\delta)}\mathbf{B} \tag{4}$$

This suggests a different definition of geometric spin:

$$s = \frac{\pi/2}{\text{Minimum angle between independent states}} \tag{5}$$

We will see that this definition of geometric spin has more physical significance than the earlier definition. The reason is that multiple values of a function might describe the same physical state (e.g. both $v = v_0$ and $v = -v_0$ yield the same value of v^2). Hence, a rotation need not return the function to its original value in order to return to its original physical state. But the rotation angle between independent states must be the same both mathematically and physically.

It is often possible to describe a physical state in different ways with different spin states. For example, temperature at a point is a scalar quantity (spin-0), but the gradient of temperature is a vector quantity (spin-1). Yet with appropriate boundary conditions, specifying the temperature gradient throughout space is equivalent to specifying the temperature itself throughout space.

A. Integer spin

First, we will discuss integer values of the geometric spin.

1. *Spin = 0*

A state with spin zero is called a "scalar". A scalar value ($\Xi_0 = a$) has a basis state $\mathbf{A} = 1$ and is independent of rotation angle. Therefore the "angle of symmetry" is infinite. When describing physical quantities, the amplitude a is real-valued and the "phase" is always zero. The amplitude of the basis state $\mathbf{A}$ is independent of rotation: $a = a_0$. There is no additional basis function $\mathbf{B}$.

2. *Spin = 1*

A sample 2-D spin-1 state $\Xi_1 = \mathbf{v} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$ has unit vectors as basis states ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), and real-valued amplitudes (a and b). An initial state with amplitudes $(a_0, 0)$ reaches the independent state with amplitudes $(0, a_0)$ after rotation by $\pi/2$ radians. An infinitesimal rotation changes the wave function $\mathbf{v}$ by $R_{d\boldsymbol{\delta}}\mathbf{v} - \mathbf{v} = d\delta\, \hat{\boldsymbol{\delta}} \times \mathbf{v}$. Rotation can also be expressed in terms of a matrix:

$$R_{d\delta}\mathbf{v} - \mathbf{v} = -\mathrm{i}\, d\delta\, \Sigma_z'^{(1)} \cdot \mathbf{v} = -\mathrm{i}\, d\delta \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \tag{6}$$

The matrix $\Sigma_z'^{(1)}$ is the "spin matrix" for 2-D vector rotation about the z-axis. We can interpret the above equation as a differential equation:

$$d\mathbf{v} = -\mathrm{i}\, d\delta\, \Sigma_z'^{(1)} \cdot \mathbf{v} \tag{7}$$

Integration yields:

$$R_\delta \mathbf{v} = \exp{(-\mathrm{i}\Sigma_z'^{(1)}\delta)} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_0 \cos\delta - b_0 \sin\delta \\ b_0 \cos\delta + a_0 \sin\delta \end{bmatrix} \tag{8}$$

This equation shows how an arbitrary initial spin-1 state transforms under rotations in two dimensions. In three dimensions, there are three orthogonal unit vectors and three 3×3 independent "spin" matrices.

$$\Sigma_x^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}; \; \Sigma_y^{(1)} = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}; \; \Sigma_z^{(1)} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

These are the generators of rotations for spin-1 states (vectors) in three dimensions. Note that these matrices define the usual cross product:

$$-\mathrm{i}\delta_j \Sigma_j^{(1)} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \delta_y a_z - \delta_z a_y \\ \delta_z a_x - \delta_x a_z \\ \delta_x a_y - \delta_y a_x \end{bmatrix} = \boldsymbol{\delta} \times \mathbf{a} \tag{10}$$

It is sometimes convenient to choose basis states so that eigenvalues of $\Sigma_z^{(1)}$ are diagonalized. This can be done by using the three basis states $(\mathbf{A}_+, \mathbf{A}_0, \mathbf{A}_-) = (-(\hat{\mathbf{x}} + \mathrm{i}\hat{\mathbf{y}})/\sqrt{2}), \hat{\mathbf{z}}, (\hat{\mathbf{x}} - \mathrm{i}\hat{\mathbf{y}})/\sqrt{2})$. The subscripts on the basis states represent the eigenvalues ($+1$, 0, and -1) associated with rotation about the z-axis. The unit imaginary may be interpreted as representing phase with respect to rotation, as in:

$$\begin{aligned} \mathbf{A}_+(t) =& \mathrm{Re}(-(\hat{\mathbf{x}} + \mathrm{i}\hat{\mathbf{y}}))\exp(-\mathrm{i}\omega t)) = -(\hat{\mathbf{x}}\cos(\omega t) + \hat{\mathbf{y}}\sin(\omega t)) \\ \mathbf{A}_-(t) =& \mathrm{Re}((\hat{\mathbf{x}} - \mathrm{i}\hat{\mathbf{y}}))\exp(-\mathrm{i}\omega t)) = (\hat{\mathbf{x}}\cos(\omega t) - \hat{\mathbf{y}}\sin(\omega t)) \end{aligned} \tag{11}$$

. The amplitudes are then $\mathbf{a} = (a_+, a_0, a_-) = (-(a_x - \mathrm{i}a_y)/2, a_z, (a_x + \mathrm{i}a_y)/2))$. The complete state is given by:

$$\Xi_1 = \mathrm{Re}(a_+\mathbf{A}_+ + a_0\mathbf{A}_0 + a_-\mathbf{A}_-) = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} \tag{12}$$

For these basis states, the spin matrices are:

$$\Sigma_x^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \; \Sigma_y^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}; \; \Sigma_z^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

As noted previously, the eigenvalues of $\Sigma_z^{(1)}$ are diagonalized so that:

$$\Sigma_z^{(1)} \begin{bmatrix} a_+ \\ a_0 \\ a_- \end{bmatrix} = \begin{bmatrix} +a_+ \\ 0 \\ -a_- \end{bmatrix} \tag{14}$$

In physical terms, the three independent states with diagonalized eigenvalues of $\Sigma_z^{(1)}$ correspond to the rotation axis ($\mathbf{A}_0$), a perpendicular axis rotating clockwise ($\mathbf{A}_-$), and a perpendicular axis rotating counterclockwise ($\mathbf{A}_+$).

3. *Spin = 2*

An example of a spin-2 state is a tensor product of two vectors: $\mathbf{u} \otimes \mathbf{v}$. Since both vectors change sign under π-radian rotation in the plane of the vectors, the "angle of symmetry" is π radians and spin is two.

A physically important example of a spin-2 system is incompressible strain. Let the state $\mathbf{A}$ consist of expansion along the x-axis and corresponding compression along the y-axis in an infinitesimal neighborhood of a point. For a square centered at the point with edges parallel to the x and y axes, this corresponds to a "normal" strain. A pure normal strain in the x-y plane has amplitude $a_0 = \partial_x \xi_x - \partial_y \xi_y$. Incompressibility requires that for this state, $\partial_x \xi_x = -\partial_y \xi_y = a_0/2$. Rotation of coordinates by angle δ to (x', y') then yields:

$$\begin{aligned} \partial_{x'}\xi_{x'} - \partial_{y'}\xi_{y'} &= (\cos\delta\,\partial_x + \sin\delta\,\partial_y)(\cos\delta\,\xi_x + \sin\delta\,\xi_y) - (\cos\delta\,\partial_y - \sin\delta\,\partial_x)(\cos\delta\,\xi_y - \sin\delta\,\xi_x) \\ &= \big((\cos\delta^2 - \sin\delta^2)(\partial_x\xi_x - \partial_y\xi_y) + 2\cos\delta\,\sin\delta(\partial_x\xi_y + \partial_y\xi_x)\big) \\ &= ((\cos(2\delta))(\partial_x\xi_x - \partial_y\xi_y) + \sin(2\delta)(\partial_x\xi_y + \partial_y\xi_x)) \end{aligned} \tag{15}$$

At $\delta = \pi/4$, the normal strain no longer depends on its initial value. Instead, the normal strain at $\pi/4$ rotation is equal to $\partial_x \xi_y + \partial_y \xi_x$. This is the initial amplitude of an independent state $\mathbf{B}$, which is the shear strain.

Figure 1 shows how these states are related.

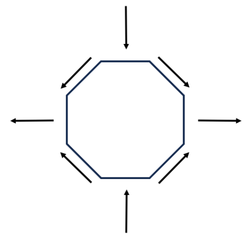

FIG. 1. Displacements associated with normal and shear strains. Rotation of $\pi/4$ changes from one to the other along fixed axes $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. The strain is defined at each point (e.g. center of the diagram) from derivatives of the displacement field in an infinitesimal neighborhood.

A general state of incompressible strain combines normal strain and shear strain:

$$\Xi_2 = a\mathbf{A} + b\mathbf{B} = (\partial_x \xi_x - \partial_y \xi_y)\mathbf{A} + (\partial_x \xi_y + \partial_y \xi_x)\mathbf{B} \tag{16}$$

In two dimensions with states $\mathbf{A}$ and $\mathbf{B}$, the spin matrix for $s = 2$ is simply twice the spin matrix for $s = 1$:

$$\Sigma_z'^{(2)} = 2\Sigma_z'^{(1)} = 2\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{17}$$

An arbitrary spin-2 state transforms under rotation as:

$$R_\delta \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \exp\left(-\mathrm{i}\Sigma_z'^{(2)}\delta\right) \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_0 \cos(2\delta) - b_0 \sin(2\delta) \\ b_0 \cos(2\delta) + a_0 \sin(2\delta) \end{bmatrix} \tag{18}$$

The principal strain angle is defined by the equation:

$$\tan(2\delta_p) = \frac{\partial_x \xi_y + \partial_y \xi_x}{\partial_x \xi_x - \partial_y \xi_y} \tag{19}$$

This determines the axis of expansion.

In three dimensions a spin-2 system has five independent states, and an arbitrary rotation requires a 5×5 matrix. With diagonalized eigenvalues of the spin matrix for rotation about the z-axis, the spin matrices are: [2]

$$\Sigma_x^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}; \; \Sigma_y^{(2)} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}; \; \Sigma_z^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{20}$$

We can envision the different m_s values as follows: A state with $m_s = \pm 2$ corresponds to the 2-D description above rotating either clockwise or counter-clockwise about the z-axis (through the page). An $m_s = 0$ state with positive amplitude corresponds in cylindrical coordinates to a state with outward displacement along the $\pm z$-axis ($\partial_z \xi_z > 0$), inward displacement along the radial axis ($\partial_r \xi_r < 0$), and azimuthal symmetry ($\partial_\phi \boldsymbol{\xi} = 0$). The x-axis in Fig. 1 is rotated to the z-axis. An $m_s = \pm 1$ state corresponds to the 2-D pattern of Fig. 1 rotated so that the z-axis is at $\pi/4$ radians (halfway between the x and y-axis in the figure), and rotating either clockwise or counterclockwise about the z-axis. This results in a change of sign under π rotation. Keep in mind that the state is defined at each point (based on derivatives of displacement in an infinitesimal neighborhood) and is unrelated to any spatial pattern of displacements over finite distances.

Gravitational waves are normally described as spin-2 disturbances of the metric tensor, but the effects of gravitational waves can be visualized as displacements like those that give rise to normal and shear strain.[3]

B. Orbital Quantum Numbers

In the discussion of spin quantum numbers above, we only considered amplitudes evaluated at a single point. Spatial or temporal derivatives may be used to define particular states (e.g. strain or wave velocity direction), but variations over finite distances or angles are not relevant to the determination of geometric spin. Extension to fields with spatial variation introduces orbital quantum numbers. If a state varies with position, then rotations cause changes in the spatial arguments of the amplitudes (or components) as well as changes in basis states (e.g. unit vectors). [4] If the arguments include an azimuthal angle, then the coefficient of the angle in the argument is the "orbital" quantum number.

For any scalar amplitude $f(\mathbf{r})$, the effect of rotation of the state is equivalent to an opposite rotation of its position argument. For an infinitesimal rotation by angle $\boldsymbol{\delta}$:

$$\begin{aligned} R_{\boldsymbol{\delta}} f(\mathbf{r}) = & \quad f(R_{\boldsymbol{\delta}}^{-1}(\mathbf{r})) \\ \approx & \quad f(\mathbf{r} - \boldsymbol{\delta} \cdot \partial_{\boldsymbol{\delta}} \mathbf{r}) \\ \approx & \quad f(\mathbf{r} + i\delta_j \Sigma_j^{(1)} \cdot \mathbf{r}) \\ \approx & f(\mathbf{r}) + i(\delta_j \Sigma_j^{(1)} \cdot \mathbf{r}) \cdot \nabla f(\mathbf{r}) \\ \approx & \quad f(\mathbf{r}) - (\boldsymbol{\delta} \times \mathbf{r}) \cdot \nabla f(\mathbf{r}) \\ \approx & \quad f(\mathbf{r}) - \boldsymbol{\delta} \cdot (\mathbf{r} \times \nabla) f(\mathbf{r}) \end{aligned} \tag{21}$$

where $\Sigma_j^{(1)}$ is the spin matrix relating different components of position $\mathbf{r}$ under rotation.

In quantum mechanics, the momentum operator is $-\mathrm{i}\hbar\nabla$. However, we will assume that any units are included in the function $f(\mathbf{r})$ so that $\mathbf{P} = -\mathrm{i}\nabla$. Whether or not this represents momentum depends on what quantity it operates on. We will simply call this the "imaginary gradient". Using this notation, the effect of rotation on the scalar amplitudes can be expressed as:

$$R_{\boldsymbol{\delta}} f(\mathbf{r}) \approx f(\mathbf{r}) - \mathrm{i}\boldsymbol{\delta} \cdot (\mathbf{r} \times \mathbf{P}) f(\mathbf{r}) \tag{22}$$

In quantum mechanics, the orbital angular momentum operator is $\mathbf{L} = \mathbf{r} \times \mathbf{P}$. Taking $\mathbf{P}$ as the imaginary gradient, we will call $\mathbf{L}$ the "imaginary angular derivative". Application to rotation yields:

$$R_{\boldsymbol{\delta}} f(\mathbf{r}) = f(\mathbf{r}) - \mathrm{i}(\boldsymbol{\delta} \cdot \mathbf{L}) f(\mathbf{r}) \tag{23}$$

With the definitions used for $\mathbf{P}$ and $\mathbf{L}$, this equation is valid for rotation of any function $f(\mathbf{r})$.

In 2-D polar coordinates $\mathbf{r} = (r, \phi)$, so the operator for L_z is:

$$L_z = rP_\phi = -\mathrm{i}\partial_\phi \tag{24}$$

where the z-axis is perpendicular to the 2-D system.

In three dimensions, the imaginary angular derivative is:

$$\mathbf{L} = -\mathrm{i}\mathbf{r} \times \nabla = -\mathrm{i}(\hat{\boldsymbol{\phi}}\partial_\theta - \hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\partial_\phi) \tag{25}$$

The Laplacian is:

$$\begin{aligned} \nabla^2 f(\mathbf{r}) &= \frac{1}{r}\partial_r^2(rf(\mathbf{r})) - \frac{1}{r^2}\mathbf{L} \cdot \mathbf{L} f(\mathbf{r}) \\ &= \frac{1}{r}\partial_r^2(rf(\mathbf{r})) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta f(\mathbf{r})) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f(\mathbf{r}) \end{aligned} \tag{26}$$

Substituting $\boldsymbol{\phi}$ for $\boldsymbol{\delta}$ in Eq. 21 yields:

$$\partial_{\boldsymbol{\phi}} f(\mathbf{r}) = -\mathbf{r} \times \nabla f(\mathbf{r}) \tag{27}$$

If we include the effect of spatial variations of the amplitudes on rotation of a vector field $\mathbf{v}(r, \phi)$, then Eq. 6 is modified to yield the complete infinitesimal rotation operator:

$$R_{d\delta}\mathbf{v}(r, \phi) - \mathbf{v}(r, \phi) = -\mathrm{i}\, d\boldsymbol{\delta} \cdot (\boldsymbol{\Sigma}^{(1)} + \mathbf{L})\mathbf{v}(r, \phi) \tag{28}$$

For finite rotations of a vector field in 2-D, an initial state with amplitudes $(a_0(r,\phi), b_0(r,\phi))$ becomes:

$$R_{d\boldsymbol{\delta}}\mathbf{v}(\mathbf{r}) = \exp\left(-\mathrm{i}\,\delta\,\Sigma_z^{\prime(1)}\right)\begin{bmatrix} a_0\left(R_\delta^{-1}(r,\phi)\right) \\ b_0\left(R_\delta^{-1}(r,\phi)\right)\end{bmatrix} = \begin{bmatrix} a_0(r,\phi-\delta)\cos\delta - b_0(r,\phi-\delta)\sin\delta \\ b_0(r,\phi-\delta)\cos\delta + a_0(r,\phi-\delta)\sin\delta\end{bmatrix} \tag{29}$$

For a spin-2 field varying with position, rotation yields:

$$R_{d\delta}\Xi_2(r,\phi) - \Xi_2(r,\phi) = -\mathrm{i}\,d\boldsymbol{\delta}\cdot(\boldsymbol{\Sigma}^{(2)} + \mathbf{L})\Xi_2(r,\phi) \tag{30}$$

Any state (or field of states in 3-D space) may be expressed as a column vector of components $f_i(\mathbf{r})$. For example, a 3-D vector field has three components $(f_x(\mathbf{r}), f_y(\mathbf{r}), f_z(\mathbf{r}))$. A 2-spinor (discussed below) has components $(f_{up}(\mathbf{r}), f_{down}(\mathbf{r}))$. Rotation changes the wave function via spin matrices that mix different components, and via rotation of the functional argument (the position $\mathbf{r}$). Since a 2π rotation returns the position to its original location, the Fourier decomposition of $f_i(r,\phi,z)$ can only depend on integer multiples of ϕ. Since $L_z = -\mathrm{i}\partial_\phi$ in these coordinates, orbital quantum numbers are integers.

Another way to see this is that at the origin ($\mathbf{r} = 0$), behavior of the wave function under 2π rotation is determined only by the eigenvalues (m_s) of the spin matrices since $\mathbf{L} = \mathbf{r}\times\mathbf{p} = 0$ at the origin. Rotation of 2π around a neighboring point must have the same effect on the wave function, so including position-dependence in the amplitudes $f_i(\mathbf{r})$ does not change the effect of a 2π rotation. This again implies that the orbital quantum numbers (m_L) must be integers, and not half-integers.

C. Half-integer spin

A spin-1/2 system returns to itself only after a 4π radian rotation, and changes by a minus sign upon a 2π rotation. This can seem quite puzzling, since rotation of the coordinate system by 2π radians changes the sign of the state variables even though the universe is unchanged. The resolution of this paradox is that physical quantities are computed from bilinear products of the state variables. This means that physical quantities are unchanged by a 2π rotation. For comparison, if $a^2 = (\cos\phi + 1)/2$, it would not be surprising to find that $a = \cos(\phi/2)$.

The significance of spin 1/2 therefore lies in the effect of rotation by π rather than rotation by 2π or 4π. As we have seen for integer spins, a phase change of $\pi/2$ results in a new state that is orthogonal to the original state. This property distinguishes the"phase" angle from the geometric angle. Hence, rotation of a spin-1/2 state by π results in a state that is linearly independent of the original state. This is not the case for systems with integer spin.

The simplest example of a spin-1/2 system is a scalar wave. Start from the equation:

$$\partial_t^2 a - c^2\partial_z^2 a = 0\,. \tag{31}$$

The derivatives can be factored as:

$$(\partial_t + c\partial_z)(\partial_t - \partial_z)a = 0\,. \tag{32}$$

The general solution consists of a superposition of forward- and backward-propagating waves:

$$f(z,t) = a_F(ct - z) + a_B(ct + z) \tag{33}$$

where $a_F(ct-z)F$ and $a_B(ct+z)$ are amplitudes of the forward- and backward-propagating waves, respectively. The value of the scalar function $f(z,t)$ is independent of rotation, but the state can change from forward-propagating to backward-propagating or vice versa under rotation by π radians. These are clearly independent states. The 1-D wave equation can be written as a first-order matrix equation for the time derivatives ($\dot{a}_B$ and $\dot{a}_F$):

$$\partial_t\begin{bmatrix}\dot{a}_B \\ \dot{a}_F\end{bmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} c\partial_z\begin{bmatrix}\dot{a}_B \\ \dot{a}_F\end{bmatrix} = 0\,. \tag{34}$$

Extension to two dimensions requires four independent states with real-valued amplitudes. Extension to three dimensions requires four independent states with complex amplitudes.

In two dimensions, which we will call the x-z plane, an arbitrary scalar wave can be defined by a real-valued four-component ”bispinor” wave function. For a scalar wave propagating along a single axis, the components of the wave function are:

$$\psi = \begin{bmatrix}\dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2}\end{bmatrix} \tag{35}$$

where $\dot{a}_{B+}$ is the magnitude of the positive contribution to the time derivative of the backward-propagating wave, $\dot{a}_{F+}$ is the magnitude of the positive contribution to the time derivative of the forward-propagating wave, $\dot{a}_{F-}$ is the magnitude of the negative contribution to the time derivative of the forward-propagating wave, and $\dot{a}_{B-}$ is the magnitude of the negative contribution to the time derivative of the backward-propagating wave.

Since rotation does not change the value of the scalar function $f(z,t)$, we could treat the positive and negative contributions independently, resulting in two independent spinors with two components each: $(\dot{a}_{F+}^{1/2}, \dot{a}_{B+}^{1/2})$ and $(\dot{a}_{F-}^{1/2}, \dot{a}_{B-}^{1/2})$. The complete four-component wave function is called a "bispinor".

For wave speed c, physical quantities are:

$$\partial_t f(z,t) = \psi^T \sigma_3 \psi = \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} \tag{36}$$

$$c\partial_z f(z,t) = \psi^T \beta^3 \psi = \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} \tag{37}$$

The matrix σ_3 is the Dirac matrix used to compute the z-component of spin in relativistic quantum mechanics. The matrix β^3 is called $-\gamma^5$ in the chiral representation, but we will see below that it should be recognized as one of three orthogonal matrices. Note that for the forward-propagating components ($F+$ and $F-$), the temporal and spatial derivatives have opposite sign. The wave velocity matrix is:

$$v_z = -c\beta^3\sigma_3 = c \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{38}$$

The equation of evolution is:

$$\partial_t \psi - c\sigma_3\beta_3\partial_z\psi = 0\,. \tag{39}$$

Multiplication from the left by $\psi^\dagger\sigma_3$ and addition of the transpose yields:

$$\begin{aligned} & \partial_t(\psi^T\sigma_3\psi) - c\partial_z(\psi^T\beta_3\psi) \\ =& \partial_t(|\dot{a}_{B+}| - |\dot{a}_{F-}| + |\dot{a}_{F+}| - |\dot{a}_{B-}|) + c\partial_z(-|\dot{a}_{B+}| - |\dot{a}_{F-}| + |\dot{a}_{F+}| + |\dot{a}_{B-}|) \\ =& (\partial_t - c\partial_z)(|\dot{a}_{B+}| - |\dot{a}_{B-}|) + (\partial_t + c\partial_z)(|\dot{a}_{F+}| - |\dot{a}_{F-}|) \\ =& \partial_t^2 f(z,t) - c^2\partial_z^2 f(z,t) = 0\,. \end{aligned} \tag{40}$$

For example, a scalar plane wave propagating in the $+z$ direction can be described by the wave function:

$$\psi = \frac{\sqrt{\omega f_0}}{2} \begin{bmatrix} 0 \\ -1 + \cos(\omega t - kz) \\ 1 + \cos(\omega t - kz) \\ 0 \end{bmatrix} \tag{41}$$

with $\omega = ck$, and ω, k, and f_0 assumed to be positive-definite. This wave function has both positive and negative contributions to the scalar wave amplitude at each point, but it does not have discontinuities that would result from strict separation of positive and negative values. Each component has the same phase of a forward-propagating wave. Each component is also real-valued, and these remain real-valued under velocity rotation in the x-z plane.

With this wave function, the time derivative of wave amplitude is:

$$\partial_t f(z,t) = \psi^\dagger\sigma_3\psi = \omega f_0 \cos(\omega t - kz) \tag{42}$$

The spatial derivative is:

$$\partial_z f(z,t) = \frac{1}{c}\psi^\dagger\beta^3\psi = -kf_0\cos(\omega t - kz) \tag{43}$$

Extension to arbitrary spatial derivative direction in 3-D utilizes the following matrices:

$$\beta^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \quad \beta^2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix} ; \quad \beta^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{44}$$

These are quite similar to the 2×2 Pauli matrices and have the same algebra $\beta^i \beta^j = \mathrm{i}\epsilon_{i,j,k}\beta^k$. The superscripts $(1, 2, 3)$ correspond to the axes (x, y, z). The gradient of the scalar function is now:

$$\nabla f = (1/c)\psi^\dagger \boldsymbol{\beta} \psi \tag{45}$$

The spin matrix for rotation about the y-axis is:

$$\Sigma_y^{(1/2)} = \frac{1}{2}\beta^2 \tag{46}$$

The operator for finite rotations in the x-z plane is:

$$\exp\left(-\mathrm{i}\Sigma_y^{(1/2)}\delta\right) = \begin{pmatrix} \cos(\delta/2) & 0 & -\sin(\delta/2) & 0 \\ 0 & \cos(\delta/2) & 0 & -\sin(\delta/2) \\ \sin(\delta/2) & 0 & \cos(\delta/2) & 0 \\ 0 & \sin(\delta/2) & 0 & \cos(\delta/2) \end{pmatrix} \tag{47}$$

Rotation of our sample wave function (Eq. 41) by $\pi/2$ about the y-axis yields:

$$\psi' = R_{\hat{\mathbf{y}}\pi/2}(\psi) = \exp\left(-\mathrm{i}\beta^2\pi/4\right)\frac{\sqrt{\omega f_0}}{2}\begin{bmatrix} 0 \\ -1 + \cos(\omega t - kR_{\hat{\mathbf{y}}\pi/2}(z)) \\ 1 + \cos(\omega t - kR_{\hat{\mathbf{y}}\pi/2}(z)) \\ 0 \end{bmatrix} = \frac{\sqrt{\omega f_0}}{2\sqrt{2}}\begin{bmatrix} -1 - \cos(\omega t - kx) \\ -1 + \cos(\omega t - kx) \\ 1 + \cos(\omega t - kx) \\ -1 + \cos(\omega t - kx) \end{bmatrix} \tag{48}$$

Note that the spatial coordinates before and after rotation are related by $z = R^{-1}_{\hat{\mathbf{y}}\pi/2}x$. The wave function ψ' describes a wave propagating in the x-direction with wave velocity specified by the matrix $-c\beta^1\sigma_3$ and satisfying the equation:

$$\partial_t \psi - c\sigma_3\beta^1\partial_x\psi = 0\,. \tag{49}$$

The equation for arbitrary wave direction is:

$$\partial_t \psi - c\sigma_3\boldsymbol{\beta}\cdot\nabla\psi = 0\,. \tag{50}$$

By multiplying $\psi^\dagger\sigma_3$ and adding the adjoint, we can obtain the second-order scalar wave equation:

$$\partial_t(\psi^\dagger\sigma_3\psi) - c\nabla\cdot(\psi^\dagger\boldsymbol{\beta}\psi) = \partial_t^2 f - c^2\nabla^2 f = 0\,. \tag{51}$$

Spatio-temporal arguments were included for the functions above only to clarify the meaning of each component. The geometric spin does not explicitly depend on these arguments, but only on the amplitude of each component at a single point. The effect of rotation on the spatial dependence is addressed through orbital quantum numbers as discussed in Sec. III B.

The above analysis for a scalar wave propagating along the z-axis is mathematically equivalent to a vector wave with polarization in the z-direction propagating along the z-axis. Extension to arbitrary vector waves in three dimensions requires three matrices $(\sigma_1, \sigma_2, \sigma_3)$ to describe an amplitude for each polarization axis.[5–7] Generalization to three dimensions not only requires three polarization axes, it also requires that rotation of coordinates changes both polarization and wave velocity. Hence, the wave velocity matrices should be $-c\beta^3\sigma_i$, which is the usual wave velocity operator for the Dirac equation (with $\beta^3 = -\gamma^5$ in the chiral representation). The matrix β^3 is used to compute divergence (as for a longitudinal wave). The other β matrices represent spatial derivatives perpendicular to the time derivative of the polarization, although they are not directly associated with unit vectors. Examples of vector shear waves will be given below.

An interesting fact is that a point in an elastic solid can rotate continuously in place, with its connection to distant points repeating only after a 4π rotation.[8–10] This behavior is closely related to the Dirac "belt trick" in which a belt twisted by 4π radians about its length can be returned to its original state without further changing the orientation of the ends. Wikipedia shows a similar analogy comparing a spinor with a Mobius band. [11] This behavior is sometimes called "spin one-half", but it is not directly related to geometric spin. The rotating point itself (or a Mobius band) returns to its original orientation after 2π rotation, and the mathematical description of positions or orientations are not changed when the coordinates are rotated by 2π radians. The connection between a point and its surroundings is a matter for topology. Geometric spin relates pointwise transformations under rotation. Of course, if one utilizes a wave description of the process, then a spin one-half wave function could be used to describe disturbances in a solid medium. In that case it is conceivable that a wave solution might include a point where the material rotates continuously in place.

IV. DYNAMIC SPIN

In quantum mechanics, the fundamental physical variable described by the Dirac equation is the density of spin angular momentum, or spin density ($\mathbf{s} = (\hbar/2)\psi^\dagger\boldsymbol{\sigma}\psi$). Classically, spin density is the vector field whose curl is equal to twice the incompressible part of the momentum density (as obtained from a Helmholtz decomposition): $\mathbf{p} = \frac{1}{2}\nabla\times\mathbf{s}$. [5–7] Applying integration by parts to the usual expression for angular momentum, $\mathbf{r}\times\mathbf{p}$, yields:

$$\begin{aligned}&\int \mathbf{r}\times\frac{1}{2}(\nabla\times\mathbf{s})d^3r = \frac{1}{2}\int\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \mathbf{r}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{r}\right)d^3r\\ &= \frac{1}{2}\int\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \partial_i(r_i\mathbf{s}) + \mathbf{s}(\nabla\cdot\mathbf{r}) - \mathbf{s}\cdot\nabla\mathbf{r}\right)d^3r\\ &= \int \mathbf{s}\,d^3r\,.\end{aligned} \tag{52}$$

The total derivatives do not contribute to the last line because they can be converted into surface integrals that are assumed to vanish.

This means that "spin" angular momentum is what we ordinarily mean by angular momentum, expressed in coordinate-independent form. It represents rotation of a medium with inertia.

It is sometimes stated that spin angular momentum refers to rotation of an object about its center, whereas orbital angular momentum refers to revolution of an object around a point other than its center. This classification is misguided. In terms of densities, "spin" refers to fixed points whereas "orbital" refers to changing positions. However, any motion of a rigid object can be described entirely in terms of spin angular momentum. Orbital angular momentum arises from torques in a material and is associated with wave propagation. Confusion results from the fact that according to modern physics, objects propagate through space as waves. Hence it is the vacuum itself that has inertia, and spin angular momentum is associated with rotational motion of the vacuum. The observed motion of objects is the result of wave propagation, which is associated with orbital angular momentum.

The rotational kinetic energy is: [7]

$$\begin{aligned}K_R &= \frac{1}{2\rho}\int \tilde{p}^2 d^3r = \frac{1}{2\rho}\int\left[\frac{1}{2}\nabla\times\mathbf{s}\right]^2 d^3r\\ &= \frac{1}{8\rho}\int\left[\mathbf{s}\cdot[\nabla\times(\nabla\times\mathbf{s})] + \nabla\cdot(\mathbf{s}\times(\nabla\times\mathbf{s}))\right]d^3r\\ &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\,d^3r,\end{aligned} \tag{53}$$

where $\mathbf{w} = \nabla\times\mathbf{u}/2$ is the instantaneous angular velocity (described earlier as "kinetic spin"). In this case the divergence term does not to contribute to the volume integral because it can be converted into a surface integral at infinity (and assumed to vanish). The rotational kinetic energy is in quadrature to the conventional kinetic energy $(1/2)\rho u^2$, but yields the same integrated total.

For a Lagrangian density dependent on motion only through kinetic energy, the spin density ($\mathbf{s}$) is the momentum conjugate to angular velocity:

$$\frac{\delta}{\delta w_i}\int\frac{1}{2}w_j s_j\,d^3r = \frac{1}{2}\int\left(\frac{\delta w_j}{\delta w_i}s_j + w_j\frac{\delta s_j}{\delta w_i}\right)d^3r = \frac{1}{2}s_i + \frac{1}{2}s_i = s_i\,, \tag{54}$$

where integration by parts was used twice to evaluate the second term in the integral.

Spin density can be used to describe rigid rotations as well. See Ref. 7 for an example.

Classical spin density should be a basic concept taught in undergraduate studies, but it has been overlooked throughout the history of physics. Instead, students have been taught that classical angular momentum depends on the coordinates, and that "intrinsic" spin angular momentum is some sort of mysterious quantum property. Both of these claims are incorrect. Classical angular momentum can easily be defined independently of coordinates as above, and that definition corresponds precisely to quantum mechanical spin angular momentum.[5–7] The simple, ordinary, interpretation of angular momentum as a measure of rotational motion of a substance with inertia is completely compatible with modern physics. The fact that the measured spin angular momentum of elementary particles is quantized does not detract from this commonsense interpretation of spin density.

People often confuse the spin quantum number with spin angular momentum. The two are only closely related because in quantum mechanics, the wave polarization described by the Dirac equation is spin density. This fact is obscured because the characteristic angular momentum $\hbar$ is factored out of the equation and applied instead to physical operators (e.g. spin density $= \psi^\dagger(\hbar\boldsymbol{\sigma}/2)\psi$). Imagine doing the same thing in classical physics: "angular momentum density $= \hbar\mathbf{r}\times\mathbf{p}$." That equation could be valid with dimensionless variables for position $\mathbf{r}$ and momentum density $\mathbf{p}$, but normally we associate the units with the variables themselves. It would be more sensible to keep $\hbar$ in the wave function and express the probability density for finding a particle at position $\mathbf{r}$ as:

$$\Pr(\mathbf{r}) = \frac{\psi^\dagger\psi}{\int \psi^\dagger\psi\, d^3r} \tag{55}$$

The spin density is then:

$$\mathbf{s} = \frac{1}{2}\psi^\dagger\boldsymbol{\sigma}\psi \tag{56}$$

and the wave function has SI units of $(\text{kg}\cdot\text{m}^2/\text{s})^{1/2}$. The Schrödinger equation can be derived as an approximation to the Dirac equation (see e.g. Ref. 12).

It is well known that elastic waves in solids have two types of momentum: "kinetic" momentum of the moving medium $(\rho\partial_t\boldsymbol{\xi})$ and "dynamic" or "elastic" momentum of the propagating wave: $\rho(\nabla\xi_j)\partial_t\xi_j$ (see e.g. Ref. 13). Clearly there must also be two types of angular momentum in an elastic solid: "spin" or "kinetic" angular momentum associated with rotation of the medium, and "orbital" or "elastic" angular momentum associated with rotation of the wave.

The velocity of the medium is $\mathbf{u} = (1/2\rho)\nabla\times\mathbf{s}$ and the angular velocity is $\mathbf{w} = (1/2)\nabla\times\mathbf{u}$. Define a vector potential $\mathbf{Q}$ by the relation $\mathbf{s} = \partial_t\mathbf{Q}$. A plausible equation of evolution is: [5–7]

$$\partial_t^2\mathbf{Q} - c^2\nabla^2\mathbf{Q} + \mathbf{u}\cdot\nabla\partial_t\mathbf{Q} - \mathbf{w}\times\partial_t\mathbf{Q} = 0. \tag{57}$$

The logic of this equation is that changes of spin density $(\partial_t\mathbf{Q})$ can only result from torque, translation, or rotation.

In terms of the bispinor wave functions, Eq. 57 becomes:

$$\partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] + c\nabla\left[\psi^\dagger\beta^3\psi\right] - \mathrm{i}c\left\{\left[\nabla\psi^\dagger\right]\times\beta^3\boldsymbol{\sigma}\psi + \psi^\dagger\beta^3\boldsymbol{\sigma}\times\nabla\psi\right\} + \mathbf{u}\cdot\nabla(\psi^\dagger\boldsymbol{\sigma}\psi) - \mathbf{w}\cdot\psi^\dagger\boldsymbol{\sigma}\psi = 0\,. \tag{58}$$

The terms correspond, in order, to those in the vector wave equation:

$$2\left(\partial_t^2\mathbf{Q} - c^2\nabla(\nabla\cdot\mathbf{Q}) + c^2\nabla\times(\nabla\times\mathbf{Q}) + \mathbf{u}\cdot\nabla\partial_t\mathbf{Q} - \mathbf{w}\times\partial_t\mathbf{Q}\right) = 0\,. \tag{59}$$

which, of course, is equivalent to Eq. 57. We assume $\nabla\cdot\mathbf{Q} = 0$ so that the waves are transverse with respect to $\mathbf{Q}$. We also assume transverse waves with $\mathbf{u}\cdot\nabla\mathbf{s} = 0$ and $\mathbf{u}\cdot\nabla\psi = 0$.

In the interest of brevity, we will simply state some results without derivation. A plausible equation of evolution of ψ is:

$$\partial_t\psi + c\beta^3\sigma_j\partial_j\psi + \frac{\mathrm{i}}{2}\bar{w}_1\beta^1\psi + \frac{\mathrm{i}}{2}w_j\sigma_j\psi = 0 \tag{60}$$

Here, the first term is the time derivative, the second term represents wave propagation, the third term represents rotation of wave velocity relative to the medium with rotation rate $\bar{w}_1$ about the spin axis, and the fourth term represents rotation of the medium with rotation rate $\mathbf{w}$.

An example solution is the shear plane wave described by the wave function:

$$\psi_{s_x,v_z} = \sqrt{\frac{\omega Q_0}{2}}\begin{bmatrix} 1 \\ \cos(\omega t - kz) - \mathrm{i}\frac{k^2Q_0}{8\rho}\sin(\omega t - kz) \\ \cos(\omega t - kz) - \mathrm{i}\frac{k^2Q_0}{8\rho}\sin(\omega t - kz) \\ 1 \end{bmatrix}. \tag{61}$$

This wave function yields spin density $s_x = (1/2)\psi^\dagger\sigma_x\psi = \omega Q_0 \cos(\omega t - kz)$. The spatial derivative is now computed with the matrix $-\beta^1$ and is proportional to displacement:

$$\xi_y = (1/2\rho)\partial_z Q_x = -(1/4c\rho)\psi^\dagger\beta^1\psi = -(kQ_0/2\rho)\cos(\omega t - kz)\,. \tag{62}$$

The wave function satisfies the full nonlinear Dirac equation in Eq. 60 with rotation rates:

$$w_x = -\frac{1}{2}\partial_z u_y = -\frac{1}{8\rho}\partial_z^2\left[\psi^\dagger\sigma_x\psi\right] = \frac{\omega k^2 Q_0}{4\rho}\cos(\omega t - kz)\,. \tag{63}$$

$$\bar{w}_1 = \frac{1}{8\rho}\partial_z^2(\psi^\dagger\beta^1\psi) = -\frac{\omega k^2 Q_0}{4\rho}\cos(\omega t - kz) = -w_x\,. \tag{64}$$

The rotation rate $\bar{w}_1$ is equal to the rate of velocity rotation about the spin axis relative to the medium. Its effect is to keep the wave velocity direction constant in spite of rotation of the medium. However, it represents rotation of the β-matrices and is not explicitly associated with any spatial direction.

For example, rotation of the wave velocity by $-\pi/2$ about the spin axis is accomplished by:

$$\psi' = \exp(\mathrm{i}\beta^1\pi/4)\psi(z\to y,t) = \frac{\sqrt{\omega Q_0}}{2}\begin{bmatrix} 1+\mathrm{i}\left(\cos(\omega t - ky) - \mathrm{i}\frac{k^2Q_0}{8\rho}\sin(\omega t - ky)\right)\\ \mathrm{i}+\left(\cos(\omega t - ky) - \mathrm{i}\frac{k^2Q_0}{8\rho}\sin(\omega t - ky)\right)\\ \mathrm{i}+\left(\cos(\omega t - ky) - \mathrm{i}\frac{k^2Q_0}{8\rho}\sin(\omega t - ky)\right)\\ 1+\mathrm{i}\left(\cos(\omega t - ky) - \mathrm{i}\frac{k^2Q_0}{8\rho}\sin(\omega t - ky)\right)\end{bmatrix}\,. \tag{65}$$

This wavefunction yields spin $s_x = \omega Q_0\cos(\omega t - ky)$ and still satisfies Eq. 60.

Now consider the physical interpretation of the terms in Eq. 60. The nonlinear terms represent rotations of wave velocity and of the medium as a whole. But they are also related to energy. Multiplying by $\mathrm{i}\psi^\dagger/2$ and adding the complex conjugate yields, with some rearranging:

$$\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi) = -\mathrm{Re}(c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi) + \frac{1}{2}\bar{w}_1\psi^\dagger\beta^1\psi + \frac{1}{2}w_j\psi^\dagger\sigma_j\psi\,. \tag{66}$$

The last term in this equation is $w_j s_j$, which is twice the rotational kinetic energy ($((k\omega Q_0)^2/4\rho)\cos^2(\omega t - kz)$ for the wave function in Eq. 61). The next to last term is proportional to the displacement $\xi_1 = (1/4c\rho)\psi^\dagger\beta^1\psi$ as in Eq. 62 (in which case the subscript "1" corresponds to the minus y-axis). The corresponding component of force density is equal to (using $\nabla\cdot\boldsymbol{\xi} = 0$):

$$f_1 = \mu\nabla^2\xi_1 = \frac{c}{4}\nabla^2\left[\psi^\dagger\beta^1\psi\right]\,. \tag{67}$$

For plane waves, the force density is proportional to displacement, so that as displacement increases from zero, the average force is half of the final force. Therefore, the conventional potential energy is

$$U = -\int \mathbf{f}\cdot d\ell = -\frac{1}{2}\mathbf{f}\cdot\boldsymbol{\xi} = -\frac{1}{32\rho}\nabla^2(\psi^\dagger\beta^1\psi)\cdot\psi^\dagger\beta^1\psi\,. \tag{68}$$

Comparing with Eqs. 64 and 66, we see that $\bar{w}_1$ is proportional to force and the next-to-last term in Eq. 66 is minus two times the conventional potential energy ($-((k\omega Q_0)^2/4\rho)\cos^2(\omega t - kz)$ for the wave function in Eq. 61), cancelling the the last term in Eq. 66 (since the rotational kinetic energy is in quadrature to the conventional kinetic energy: i.e. $\sin^2 \leftrightarrow \cos^2$).

The terms in Eq. 66 correspond to different energies as follows:

$$\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi) = -\mathrm{Re}(c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi) + \frac{1}{2}\bar{w}_1\psi^\dagger\beta^1\psi + \frac{1}{2}w_j\psi^\dagger\sigma_j\psi \tag{69a}$$

$$\mathcal{E} \quad = \quad \mathbf{P}\cdot c\hat{\mathbf{v}} \quad + \quad \mathbf{f}\cdot\boldsymbol{\xi} \quad + \quad \mathbf{w}\cdot\mathbf{s} \tag{69b}$$

where $\mathbf{P}$ is the wave momentum density and $\mathcal{E}$ is the total energy density. For the plane waves of Eq. 61, we have $\mathcal{E} = \mathbf{P}\cdot c\hat{\mathbf{v}} = (k\omega Q_0)^2/8\rho$. The last two terms cancel for plane waves.

Rotational potential energy density can be defined as $U_R = \mathbf{P}\cdot c\hat{\mathbf{v}} - U$. The rotational potential energy density (U_R) and rotational kinetic energy density (K_R) are in quadrature to their usual expressions. For the bispinor representation of plane waves, $\mathbf{P}\cdot c\hat{\mathbf{v}}$ is equal to the total energy, which has the same integrated value as twice the potential energy.

The different energy expressions are therefore:

$$K_R = \frac{1}{2}\mathbf{w}\cdot\mathbf{s} = \frac{1}{2}(\frac{1}{8\rho}\nabla\times\nabla\times\psi^\dagger\boldsymbol{\sigma}\psi)\cdot\frac{1}{2}\psi^\dagger\boldsymbol{\sigma\psi} \tag{70a}$$

$$U_R = \mathbf{P}\cdot c\hat{\mathbf{v}} + \frac{1}{2}\mathbf{f}\cdot\boldsymbol{\xi} = -\mathrm{Re}(\psi^\dagger c\beta^3\sigma_j \mathrm{i}\partial_j\psi) + \frac{1}{2}\nabla^2(\frac{1}{8\rho}\psi^\dagger\beta^1\psi)\cdot\frac{1}{2}\psi^\dagger\beta^1\psi\,. \tag{70b}$$

For plane waves we also have total energy of $\mathcal{E} = \mathrm{Re}(\psi^\dagger \mathrm{i}\partial_t\psi)$. This is the same result as for quantum mechanics.

A. Lagrangian and Hamiltonian

Having a first-order equation of evolution enables the use of variational methods. Interpreting Eq. 60 as an Euler-Lagrange equation requires distinction between terms containing one factor each of ψ and $\psi^\dagger$ or their derivatives, and terms containing two such factors. Just as spin density had to be regarded as functionally dependent on angular velocity in Eq. 54, angular velocity ($\mathbf{w}$ or $\bar{\mathbf{w}}$) should be regarded as functionally dependent on the wave function. Since we are only considering plane waves in this paper, we assume the convection term $\mathbf{u}\cdot\nabla\psi$ to be zero.

Treating ψ and $\psi^\dagger$ as independent variables, we construct a Lagrangian density $\mathcal{L} = 0$ so that terms linear in ψ and its derivatives have coefficient of one as in Eq. 60, and the two rotation terms are cut in half:

$$\mathcal{L} = \frac{1}{2}\left\{\psi^\dagger \mathrm{i}\partial_t\psi - \mathrm{i}\partial_t\psi^\dagger\psi + c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi - c\mathrm{i}\partial_j\psi^\dagger\beta^3\sigma_j\psi\right\} - \frac{1}{4}\bar{w}_1\psi^\dagger\beta^1\psi - \frac{1}{4}w_j\psi^\dagger\sigma_j\psi\,. \tag{71}$$

The Euler-Lagrange equation is:

$$\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^\dagger)} + \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\psi^\dagger)} - \frac{\partial\mathcal{L}}{\partial\psi^\dagger} = 0 \tag{72}$$

Application to Eq. 71 yields Eq. 60.

The conjugate momentum to the field ψ is p_ψ:

$$p_\psi = \frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]} = \frac{\mathrm{i}}{2}\psi^\dagger\,. \tag{73}$$

and similarly for $p_{\psi^\dagger}$.

The Hamiltonian is a special case (T^0_0) of the energy-momentum tensor: [14]

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\psi^\dagger\right]}\partial_\nu\psi^\dagger + \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\psi\right]}\partial_\nu\psi - \mathcal{L}\delta^\mu_\nu\,. \tag{74}$$

The real-valued Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left\{-c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi + c\mathrm{i}\partial_j\psi^\dagger\beta^3\sigma_j\psi\right\} + \frac{1}{4}\bar{w}_1\psi^\dagger\beta^1\psi + \frac{1}{4}w_j\psi^\dagger\sigma_j\psi\,. \tag{75}$$

Comparing with Eq. 69, we have:

$$\mathcal{H} = U_R + K_R \tag{76}$$

In the Lagrangian, the kinetic energy term is negative. Therefore, the conjugate momenta computed from the Lagrangian should include a minus sign. The dynamical (or wave) momentum density P_i is

$$P_i = -T^0_i = -\frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi^\dagger\right]}\partial_i\psi^\dagger - \frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]}\partial_i\psi = -\frac{1}{2}\mathrm{Re}\left\{\psi^\dagger\mathrm{i}\partial_i\psi\right\}\,. \tag{77}$$

The wave angular momentum density is likewise

$$\mathbf{L} = -\frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi^\dagger\right]}\partial_{\boldsymbol{\varphi}}\psi^\dagger - \frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]}\partial_{\boldsymbol{\varphi}}\psi = -\mathrm{i}\psi^\dagger\partial_{\boldsymbol{\varphi}}\psi + c.c. = -\frac{\mathrm{i}}{2}\psi^\dagger\frac{\partial r_i}{\partial\boldsymbol{\varphi}}\partial_i\psi + c.c. = -\mathrm{Re}\left\{\mathbf{r}\times\psi^\dagger\mathrm{i}\nabla\psi\right\}\,. \tag{78}$$

This expression assumes a particular origin for the axis of rotation of the angle $\boldsymbol{\varphi}$, in contrast to the coordinate-independent spin angular momentum.

For densities of total momentum ($\mathbf{P}_T$) and angular momentum ($\mathbf{J}$), we must combine the wave and medium contributions:

$$\mathbf{P}_T = \mathbf{P} + \mathbf{p} = -\mathrm{Re}\left\{\psi^\dagger \mathrm{i}\nabla\psi\right\} + \frac{1}{2}\nabla \times \psi^\dagger \frac{\boldsymbol{\sigma}}{2}\psi \,; \tag{79}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{s} = -\mathrm{Re}\left\{\mathbf{r} \times \psi^\dagger \mathrm{i}\nabla\psi\right\} + \psi^\dagger \frac{\boldsymbol{\sigma}}{2}\psi \,. \tag{80}$$

The intrinsic or conjugate total momentum density ($\mathbf{p}$) is seldom mentioned, but it can also be derived from the symmetrized Belinfante-Rosenfeld energy-momentum tensor. [15–17]

Now the imaginary gradient (operator for $\mathbf{P}$) and imaginary angular derivative (operator for $\mathbf{L}$) do represent momentum and angular momentum, respectively because of the physical interpretation of the wave function ψ. The orbital term ($\mathbf{L}$) is a form of angular momentum, but it does NOT represent motion. Rather it is derived from stresses in the medium. This orbital angular momentum can be expressed as $\mathbf{r} \times \mathbf{P}$, but in this case $\mathbf{P}$ represents the dynamic or "wave" momentum, not the intrinsic or conjugate momentum of the medium.

The eigenvalues of one component of orbital angular momentum (e.g. L_z) must be an integer, and eigenvalues of the spin matrices (e.g. $\sigma_3/2$) are $\pm 1/2$ for waves, so the eigenvalues of any component of total angular momentum ($m_L + m_s$) may be integer or half-integer. For eigenfunctions, we therefore have quantized angular momentum $J_z = (m_L + m_s)\int \psi^\dagger\psi \, d^3r$.

It is important to recognize that the momentum we normally attribute to moving objects is actually wave momentum. This fact is well-established in quantum mechanics. Objects move through space as waves, with the laws of special relativity attributable to Lorentz-invariance of the wave equation.[18] The "spin" angular momentum of elementary particles has the same form as the intrinsic angular momentum of an elastic solid moving under the influence of propagating waves. This observation refutes the common claim that "the aether, if it exists, is not detectable."

Thus, we see that the angular momentum is indeed closely linked to the spin and orbital quantum numbers, but only because the polarization vector associated with the Dirac wave function is in fact the spin density.

V. CONCLUSIONS

The term "spin" has three different meanings in physics. When describing "spinning" objects, it refers to angular velocity. We call this "kinetic" spin. With regard to mathematical functions, spin quantum numbers are the eigenvalues of rotation matrices for the basis states at each point. We call this "geometric" spin. Examples of integer spin are scalars (spin 0), vectors (spin 1), and the strain tensor (spin 2). A simple example of half-integer spin is a scalar plane wave. Orbital quantum numbers are eigenvalues of rotation of coordinates. Since the position coordinate repeats under rotation by 2π radians, orbital quantum numbers are integers. As a form of angular momentum, spin refers to the ordinary "intrinsic" or "conjugate" angular momentum of a moving inertial medium, as opposed to the "dynamic" or "orbital" angular momentum associated with wave propagation. The spin density, in both classical and quantum physics, is equal to the field whose curl is twice the momentum density.

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