Relativistic Wave Mechanics for Undergraduates

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Hypothesis

We should teach students the Dirac equation prior to the Schrödinger equation.

Why Not Schrödinger Equation?

- It is not consistent with Special Relativity.
- It does not allow for spin angular momentum.
- It is not based on classical physics.

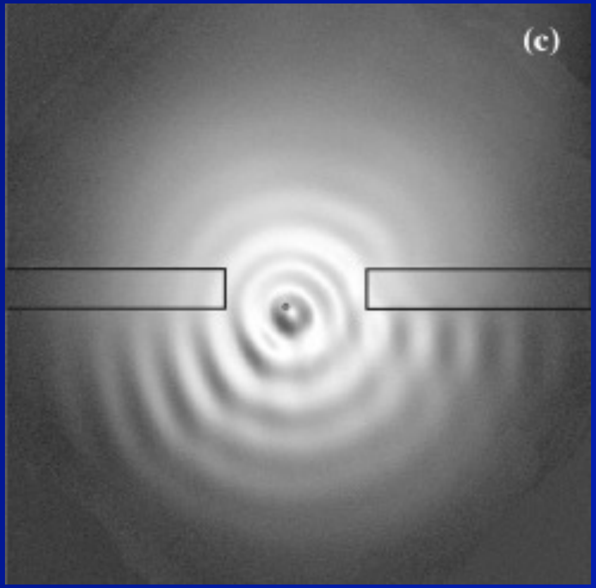
A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding.

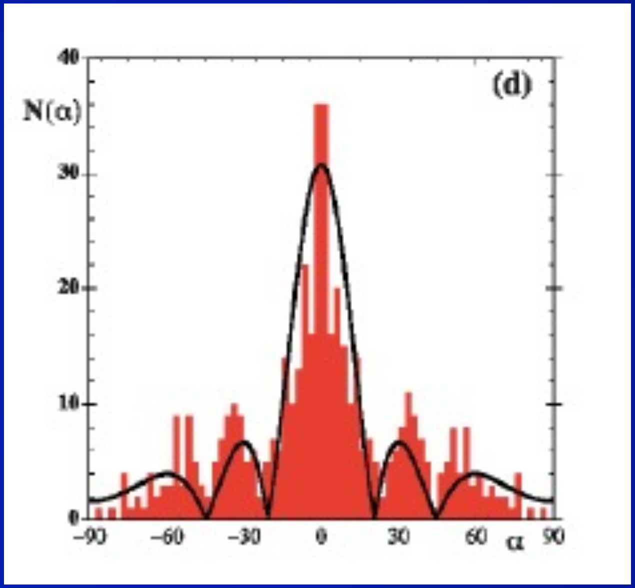
- Isaac Newton

Why Dirac Equation?

- It is Lorentz invariant.
- It incorporates spin angular momentum.
- It is easily derived from classical physics:
  1. Define spin density
  2. Find wave equation of evolution
  3. Factor the wave equation
- Schrodinger equation approximates one component of the Dirac equation.

Classical Physics = Quantum Physics





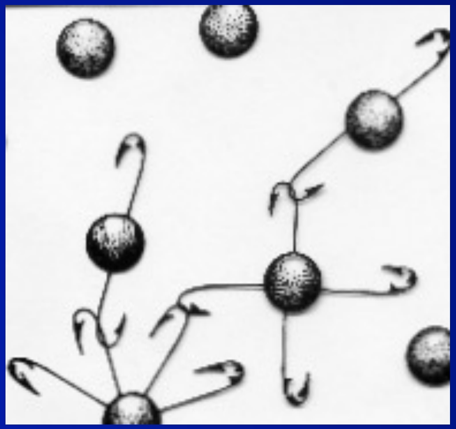
Couder & Fort , PRL **97,** 154101 (2006)

Quantum phenomena can be produced using classical hydrodynamical “pilot waves”.

See: https://thales.mit.edu/bush/index.php/4801-2/

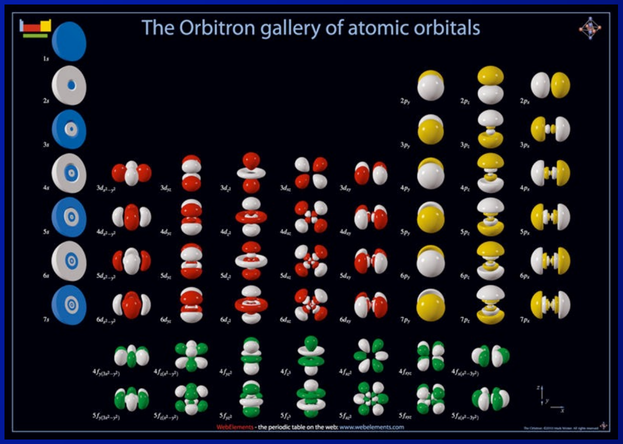
Is the Vacuum an "Aether"?

But what if these molecules, indestructible as they are, turn out to be not substances themselves, but mere affections of some other substance? - James Maxwell



https://www.chem1.com/

To deny the ether is ultimately to assume that empty space has no physical qualities whatever. The fundamental facts of mechanics do not harmonize with this view.
– Albert Einstein



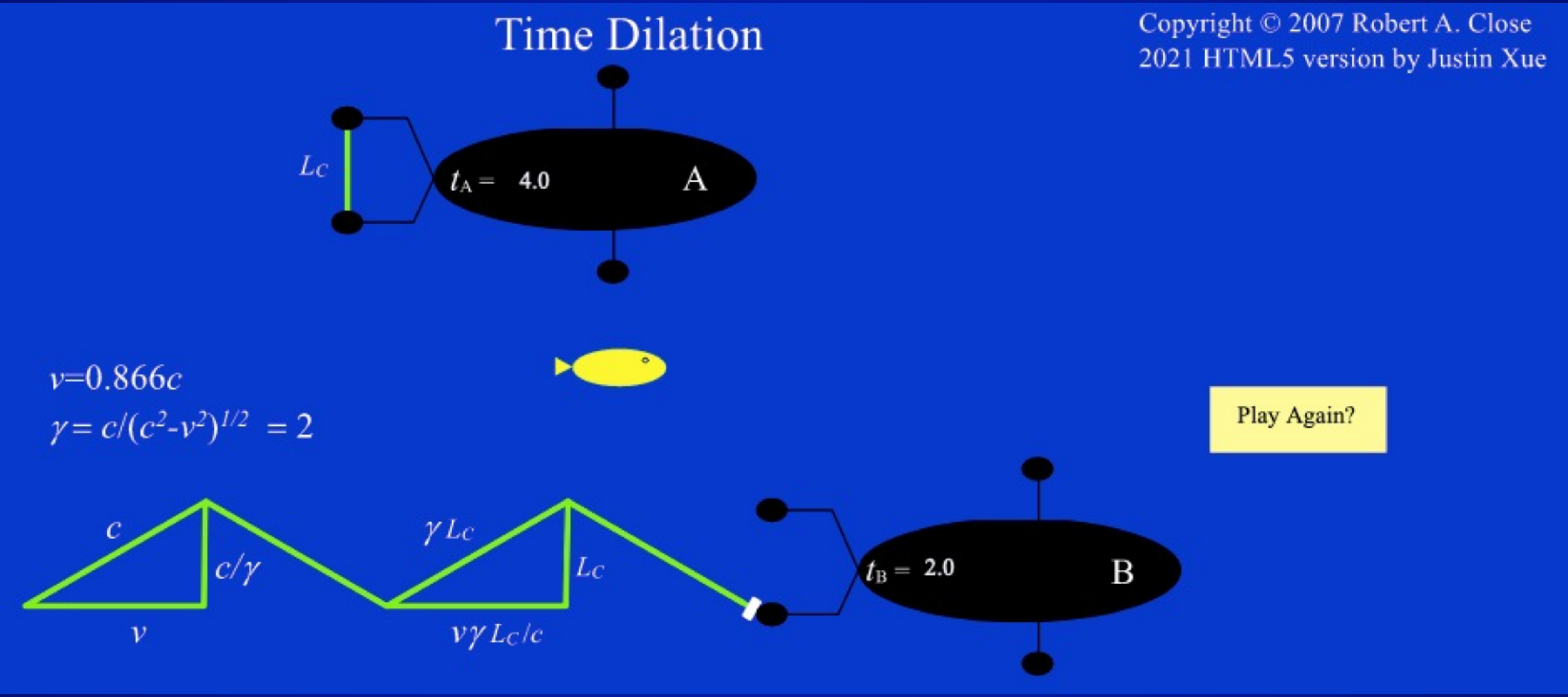
http://winter.group.shef.ac.uk/orbitron/

Is the Vacuum an "Aether"?

... the aether is no longer ruled out by relativity, and good reasons can now be advanced for postulating an aether.
- Paul Dirac 1951

The modern concept of the vacuum of space, confirmed by everyday experiment, is a relativistic ether. But we do not call it this because it is taboo. - Robert Laughlin, 2005

Is the Vacuum an "Aether"?



https://www.classicalmatter.org/UnderwaterRelativity.htm

Spin Density: Incompressible Motion

Helmholtz Decomposition of momentum density:

$$\mathbf{p} \equiv \rho\mathbf{u} = \mathbf{p}_0 + \nabla\Phi + \frac{1}{2}\nabla \times \mathbf{s}$$

Incompressible and in “rest” frame:

$$\nabla\Phi = 0,\ \mathbf{p}_0 = 0$$

$$\nabla \cdot \mathbf{u} = 0 \rightarrow \rho = \text{constant}$$

$$\nabla \cdot \rho\mathbf{u} = \nabla \cdot \mathbf{p} = 0$$

$$\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s}$$

Spin Density

$$\mathbf{p} = \frac{1}{2}\nabla \times \boldsymbol{s}$$

Spin angular momentum density is the vector field whose curl is equal to twice the incompressible momentum density ($\mathbf{p}=\rho\mathbf{u}$).* Every physics student should learn this fundamental definition of spin density.

*R. A. Close, "A classical Dirac bispinor equation," in *Ether Space-time & Cosmology, vol. 3*, edited by M. C. Duffy and J. Levy (Apeiron, Montreal, 2009).
See also Publications at the end of this slide show.

Integrated Spin Density

$$\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s};\ \mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v}$$

$$\mathbf{S} = \int \mathbf{r} \times \mathbf{p}\, \boldsymbol{d^3r} = \int \mathbf{s}\, \boldsymbol{d^3r} + \mathrm{b.t.}$$

$$K = \int \frac{1}{2}\rho v^2\, d^3r = \int \frac{1}{2}\mathbf{w} \cdot \mathbf{s}\, \boldsymbol{d^3r} + \mathrm{b.t.}$$

Spin density yields the same total angular momentum and rotational kinetic energy as conventional calculations.

Spin Density: Conjugate Momentum

Two types of momentum:
"intrinsic" & "wave"
Two types of angular momentum:
"intrinsic" (**s**) & "wave" (**L**).

$$\frac{\delta}{\delta \mathbf{w}} \int \frac{\rho v^2}{2}\, d^3 r = \frac{\delta}{\delta \mathbf{w}} \int \frac{\mathbf{w} \cdot \mathbf{s}}{2}\, \boldsymbol{d^3 r} = \int \mathbf{s}\, d^3 r$$

For Lagrange density $\mathfrak{L} = \frac{1}{2} \rho v^2$,
s is the momentum conjugate to **w**.
L derives from potential energy & torque.

Equation of Evolution

Momentum density:

$$\partial_t \mathbf{p} + \mathbf{u} \cdot \nabla \mathbf{p} = \mathbf{F}$$

Spin density (Helmholtz decomposition):

$$\partial_t \mathbf{s} + \frac{1}{\pi} \nabla \times \int \frac{\mathbf{w}' \times \mathbf{p}'}{|\mathbf{r} - \mathbf{r}'|} d^3 r = \boldsymbol{\tau}$$

Equation of Evolution

Elastic solid model: torque is derived from a vector potential **Q**:

$$\mathbf{s} \equiv \partial_t \mathbf{Q} \quad \text{and} \quad \boldsymbol{\tau} = c^2 \nabla^2 \mathbf{Q}$$

$$\partial_t^2 \mathbf{Q} - c^2 \nabla^2 \mathbf{Q} = -\frac{1}{\pi} \nabla \times \int \frac{\mathbf{w}' \times \mathbf{p}'}{|\mathbf{r} - \mathbf{r}'|} d^3 r$$

Nonlinear equation ⇒ Quantization

Simple wave equation w/o nonlinear term

Factor the Wave Equation

- 1-D Wave Equation:

$$\partial_t^2 a - c^2 \partial_z^2 a = 0$$

- Solution is Forward + Backward waves:

$$a(z,t) = a_F(z - ct) + a_B(z + ct)$$

- Independent solutions are 180° apart. Wave solutions form a spin one-half system (3-D requires bispinors).

Backward Forward

Factor the Wave Equation

$$\dot{a}(z,t) = \dot{a}_F(z-ct) + \dot{a}_B(z+ct)$$

- First-order matrix equation:

$$\partial_t \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} - c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} = 0$$

- Equivalent to second-order wave equations:

$$\partial_t^2 a_B - c^2 \partial_z^2 a_B = 0$$
$$\partial_t^2 a_F - c^2 \partial_z^2 a_F = 0$$

Factor the Wave Equation

- To accommodate rotation, separate positive and negative time derivative components:

$$\begin{aligned}\partial_t a(z,t) = &\ \dot{a}_{F+}(z-ct) \\ &- \dot{a}_{F-}(z-ct) \\ &+ \dot{a}_{B+}(z+ct) \\ &- \dot{a}_{B-}(z+ct)\end{aligned}$$

Define the wave function (chiral representation):

$$\psi(z,t) = \begin{bmatrix} (\dot{a}_{B+})^{1/2} \\ (\dot{a}_{F-})^{1/2} \\ (\dot{a}_{F+})^{1/2} \\ (\dot{a}_{B-})^{1/2} \end{bmatrix}$$

Factor the Wave Equation

- The first-order 1-D wave equation:

$$\partial_t(\psi^T\sigma_{\boldsymbol{z}}\psi) + c\partial_z(\psi^T\gamma^5\psi) = 0$$

where:

$$\psi^T\sigma_{\boldsymbol{z}}\psi = \begin{bmatrix}(\dot{a}_{B+})^{1/2}\\(\dot{a}_{F-})^{1/2}\\(\dot{a}_{F+})^{1/2}\\(\dot{a}_{B-})^{1/2}\end{bmatrix}^T \begin{pmatrix}1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -1\end{pmatrix} \begin{bmatrix}(\dot{a}_{B+})^{1/2}\\(\dot{a}_{F-})^{1/2}\\(\dot{a}_{F+})^{1/2}\\(\dot{a}_{B-})^{1/2}\end{bmatrix} = \partial_t a$$

$$\psi^T\gamma^5\psi = \begin{bmatrix}(\dot{a}_{B+})^{1/2}\\(\dot{a}_{F-})^{1/2}\\(\dot{a}_{F+})^{1/2}\\(\dot{a}_{B-})^{1/2}\end{bmatrix}^T \begin{pmatrix}-1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} \begin{bmatrix}(\dot{a}_{B+})^{1/2}\\(\dot{a}_{F-})^{1/2}\\(\dot{a}_{F+})^{1/2}\\(\dot{a}_{B-})^{1/2}\end{bmatrix} = -c\partial_z a$$

Dirac Equation

These definitions yield (for $a = Q_z/2$):

$$s_z = \partial_t Q_z = \psi^\dagger \sigma_{\mathbf{z}} \psi/2$$
$$c\partial_z Q_z = -\psi^\dagger \gamma^5 \psi/2$$

1-D wave equation:

$$0 = \{\psi^T \sigma_{\mathbf{z}} \partial_t \psi + \psi^T \gamma^5 \sigma_{\mathbf{z}} \sigma_{\mathbf{z}} c \partial_z \psi\} + \mathrm{Transp.}$$

3-D wave equation (replace z with i or j):

$$0 = \partial_t[\psi^\dagger \boldsymbol{\sigma} \psi] + c\nabla \cdot [\psi^\dagger \gamma^5 \psi]$$
$$-ic[\nabla\psi^\dagger \times \boldsymbol{\sigma}\psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla\psi]$$

Dirac Equation

Where:

$$\mathbf{s} = \partial_t \mathbf{Q} = \frac{1}{2}\psi^\dagger \boldsymbol{\sigma}\psi$$

$$c\nabla \cdot \mathbf{Q} = -\frac{1}{2}\psi^\dagger \gamma^5 \psi$$

$$c^2\nabla \times \nabla \times \mathbf{Q}$$
$$= \frac{-ic}{2}[\nabla\psi^\dagger \times \boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma} \times \nabla\psi]$$

$$0 = \frac{-ic}{2}\nabla \cdot [\nabla\psi^\dagger \times \boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma} \times \nabla\psi]$$

Dirac Equation

Electron:

$$\partial_t \psi + c\,\gamma^5 \sigma_j \partial_j \psi + iM\gamma^0 \psi = 0$$

Multiply $\psi^\dagger \sigma_i$ and add adjoint:

$$\psi^\dagger \sigma_i \partial_t \psi + \psi^\dagger c\,\gamma^5 \sigma_i \sigma_j \partial_j \psi + adj. = 0$$

Spin density equation (multiply $\hbar/2$)

$$0 = \partial_t [\psi^\dagger \boldsymbol{\sigma} \psi] + c\nabla \cdot [\psi^\dagger \gamma^5 \psi]$$

$$-ic[\nabla \psi^\dagger \times \boldsymbol{\sigma} \psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla \psi]$$

Elastic solid equation is identical.

Dirac Equation

Dynamical quantities are equivalent to those of relativistic quantum mechanics:

$$\mathbf{P} = -\left[\psi^{\dagger} i \nabla \psi\right] + \frac{1}{2} \nabla \times \left[\psi^{\dagger} \frac{\boldsymbol{\sigma}}{\mathbf{2}} \psi\right]$$

$$\mathbf{J} = -\mathbf{r} \times \left[\psi^{\dagger} i \nabla \psi\right] + \left[\psi^{\dagger} \frac{\boldsymbol{\sigma}}{\mathbf{2}} \psi\right]$$

The momentum of the medium is necessary for a symmetric energy-momentum tensor compatible with General Relativity. [Ohanian 1986]

Plane Waves

Vector plane wave:

$$\mathbf{Q}(x, y, z, t) = \hat{z} Q_0 \sin(\omega t - kz)$$

$$\mathbf{s} = \partial_t \mathbf{Q} = \hat{z} \omega Q_0 \cos(\omega t - kz)$$

Dirac representation:

$$\psi = \sqrt{2\omega Q_0} \exp\left[-\mathrm{i}\left(\omega t - kz\right)/2\right] \begin{bmatrix} 0 \\ \sin\left(\left[\omega t - kz\right]/2\right) \\ \cos\left(\left[\omega t - kz\right]/2\right) \\ 0 \end{bmatrix}$$

Note that Dirac phase is half of vector phase.

Plane Waves

TABLE I. Spin and wave velocity operators

Rotation Axis:	Initial	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	
Rotation Operator:	None	$e^{-i\sigma_1\pi/4}$	$e^{-i\sigma_2\pi/4}$	$e^{-i\sigma_3}$
Change of variable:	None	$z \to -y$	$z \to x$	z –
Final Spin Axis:	$\sigma_3\hat{\mathbf{z}}$	$-\sigma_2\hat{\mathbf{y}}$	$\sigma_1\hat{\mathbf{x}}$	
Wave	$\gamma^6\sigma_3\hat{\mathbf{x}}$	$\gamma^6\sigma_2\hat{\mathbf{x}}$	$\gamma^5\sigma_1\hat{\mathbf{x}}$	γ^0
Velocity	$-\gamma^0\sigma_3\hat{\mathbf{y}}$	$-\gamma^5\sigma_2\hat{\mathbf{y}}$	$-\gamma^0\sigma_1\hat{\mathbf{y}}$	γ^6
Operators	$\gamma^5\sigma_3\hat{\mathbf{z}}$	$-\gamma^0\sigma_2\hat{\mathbf{z}}$	$-\gamma^6\sigma_1\hat{\mathbf{z}}$	γ^5

$\sigma_3 = i\sigma_1\sigma_2$ $\qquad$ $\gamma^6 = i\gamma^0\gamma^5$

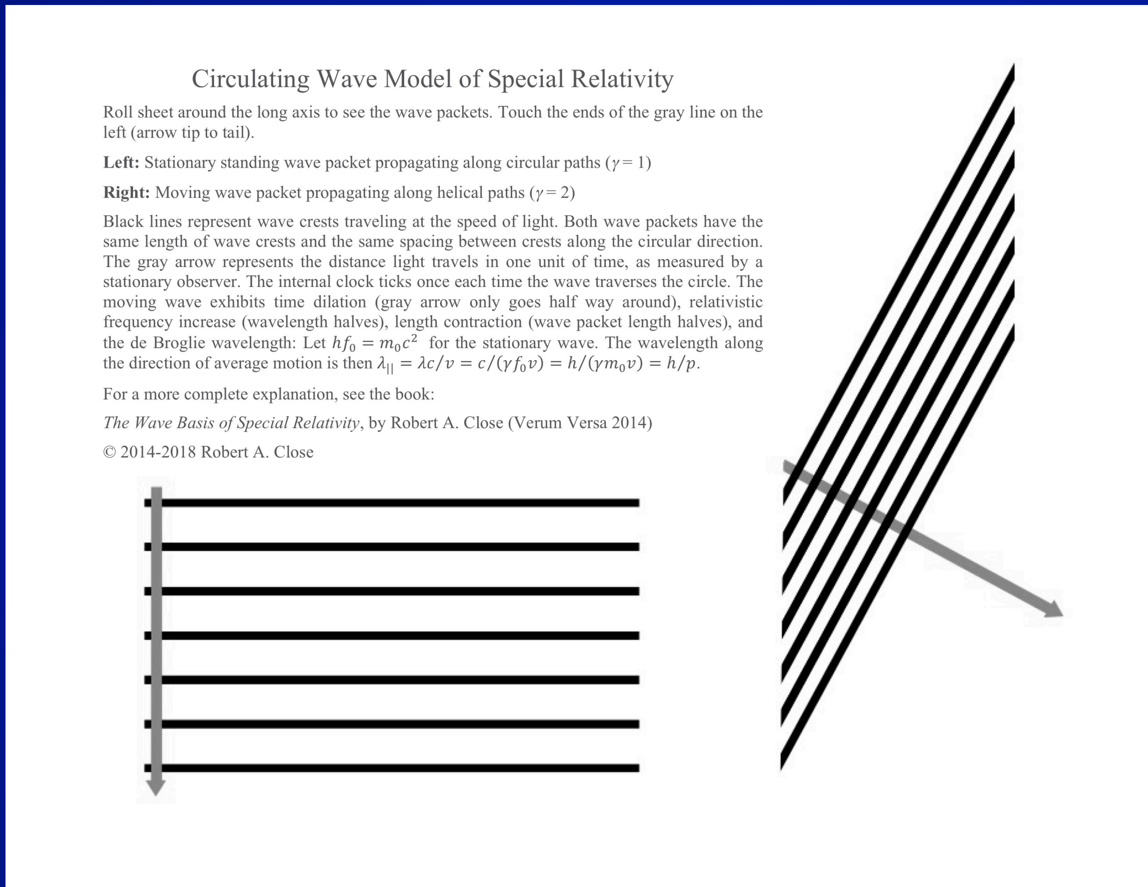
Plane Waves

The matrices $(\gamma^0, \gamma^5, \gamma^6)$ represent three orthogonal directions relative to the wave velocity.

The mass term $iM\gamma^0\psi$ represents differential rotation of wave velocity.

Hence an elementary particle can be visualized as a standing wave with waves propagating in circles.

Wave Model of Special Relativity



Circulating Wave Model of Special Relativity

Roll sheet around the long axis to see the wave packets. Touch the ends of the gray line on the left (arrow tip to tail).

Left: Stationary standing wave packet propagating along circular paths ($\gamma = 1$)

Right: Moving wave packet propagating along helical paths ($\gamma = 2$)

Black lines represent wave crests traveling at the speed of light. Both wave packets have the same length of wave crests and the same spacing between crests along the circular direction. The gray arrow represents the distance light travels in one unit of time, as measured by a stationary observer. The internal clock ticks once each time the wave traverses the circle. The moving wave exhibits time dilation (gray arrow only goes half way around), relativistic frequency increase (wavelength halves), length contraction (wave packet length halves), and the de Broglie wavelength: Let $hf_0 = m_0c^2$ for the stationary wave. The wavelength along the direction of average motion is then $\lambda_{||} = \lambda c/v = c/(\gamma f_0 v) = h/(\gamma m_0 v) = h/p$.

For a more complete explanation, see the book:

The Wave Basis of Special Relativity, by Robert A. Close (Verum Versa 2014)

Available at
www.classicalmatter.org

Spatial Reflection

Spatial reflection should invert $(\gamma^0, \gamma^5, \gamma^6)$.
Conventional parity operator fails:

$$P\psi(\mathbf{r}, t) = \gamma^0 \sigma_s \psi(-\mathbf{r}, t)$$

Longitudinal waves have zero curl. Try:

$$P\psi(\mathbf{r}, t) = \gamma^5 \sigma_s \psi(-\mathbf{r}, t)$$

where "s" indicates the spin direction. For (approximate) spin eigenfunctions:

$$P'\psi(\mathbf{r}, t) = \gamma^5 \psi(-\mathbf{r}, t)$$

Compare Feynman-Stuckelberg positron:

$$\psi_+(E; \mathbf{r}, t) = PCT\psi_-(-E; \mathbf{r}, t)$$
$$= \gamma^5 \psi_-(-E; -\mathbf{r}, -t)$$

Mirror Images

- Vector spherical harmonics:
  - Distinct for odd integers.
  - Identical for even integers.
- Dirac (half the phase of vectors):
  - Distinct for half-integers.
  - Identical for whole integers.
- Standard Model:
  - Fermions have distinct antiparticles.
  - Bosons have identical antiparticles (except $W^{+/-}$).

Exclusion Principle & Potentials

- Wave superposition:

$$[\psi_A + \psi_B]^\dagger \boldsymbol{\sigma} [\psi_A + \psi_B] = \psi_A^\dagger \boldsymbol{\sigma} \psi_A + \psi_B^\dagger \boldsymbol{\sigma} \psi_B$$
$$+\psi_A^\dagger \boldsymbol{\sigma} \psi_B + \psi_B^\dagger \boldsymbol{\sigma} \psi_A$$

- Interference terms cancel for "independent" particles. For eigenfunctions this yields the exclusion principle (anti-commutation):

$$\psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A = 0$$

- Potentials are phase shifts introduced to maintain zero interference.

Classical Dirac Equation

- Lorentz-invariant equation.
- Mass represents rotation of wave velocity.
- Spin is the angular momentum of the medium.
- QM momentum & angular momentum operators.
- Analogues of fermions and bosons.
- Spatial reflection yields “antiparticles”.
- Exclusion principle and interaction potentials.
- *Magnetic flux quantization.*
- *Interpretation of electric charge.*
- *Wave uncertainty relations.*
- *“Gravity” is wave refraction.*

Publications

- Torsion Waves in Three Dimensions: Quantum Mechanics With a Twist," **Found. Phys. Lett.** 15(1):71-83, Feb. 2002.
- “A Classical Dirac Bispinor Equation,” in *Ether Space-time & Cosmology, vol. 3,* eds. M. C. Duffy & J. Levy, (Aperion, Montreal, 2009).
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- “The Mirror Symmetry of Matter and Antimatter,” **Adv. App. Clifford Algebras 21**:283-295, 2010.
- *The Wave Basis of Special Relativity* (Verum Versa, 2014)
- “Spin Angular Momentum and the Dirac Equation,” **Electr. J. Theor. Phys. 12**(33):43-60, 2015

- More at: www.ClassicalMatter.org

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