Introduction to wave mechanics: Interactions

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The first-order quantum mechanical Dirac equation is interpreted as a representation of a second-order vector wave equation for spin angular momentum density (or spin density). This interpretation offers students a simple classical physics interpretation of relativistic quantum mechanics. This paper outlines how a classical wave theory of spin density can be used to describe particle-like waves and their interactions, offering students a conceptual bridge between classical physics and quantum mechanics. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials. Classical interpretations of magnetic flux quantization and the Coulomb potential are presented. A classical version of the single-electron Lagrangian for quantum electrodynamics is also presented.

Keywords: classical interpretation, Dirac equation, elastic solid, magnetic flux quantization, quantum electrodynamics, quantum mechanics pedagogy, spin angular momentum, spin density, teaching quantum mechanics, wave mechanics
1. INTRODUCTION

Previous work has proposed that physics students should be introduced to quantum mechanics using the Dirac equation, which is both relativistic and describes spin angular momentum, rather than via the Schrödinger equation. Classical spin density ($s$) is the field whose curl is equal to twice the momentum density ($p$). The relationship to Dirac wave functions is: [1–3]

\[ s = \partial_t Q \equiv \frac{1}{2} \left[ \psi \hat{\sigma} \psi \right] ; \]  
\[ c \nabla \cdot Q \equiv -\frac{1}{2} \left[ \psi \hat{\sigma}_5 \psi \right] ; \]  
\[ c^2 \{ \nabla \times \nabla \times Q \} \equiv -i c^2 \{ \nabla \psi \hat{\sigma}_5 \psi + \psi \hat{\sigma}_5 \psi \times \nabla \psi \} ; \]  
\[ 0 = i c^2 \nabla \cdot \{ \nabla \psi \hat{\sigma}_5 \psi + \psi \hat{\sigma}_5 \psi \times \nabla \psi \} . \]

The Dirac formalism, in addition to quantum mechanical application, has a variety of applications in describing classical physics. [1–11] There has also been observation of quantum mechanical behavior of classical systems. Quantum statistics have been observed in experiments using silicone droplets bouncing on a vibrating water tank. [12–18] Such experiments provide a physical realization of Bohmian mechanics, or pilot-wave theory. [19–21]

Wave interactions give rise to the Pauli exclusion principle and electromagnetic potentials. We discuss magnetic flux quantization and the relationship between magnetic flux and electric charge. Finally, we relate the classical Lagrangian to that of quantum electrodynamics.

2. WAVE INTERACTIONS

Suppose we have two Dirac wave functions $\psi_A$ and $\psi_B$, representing particle-like waves $A$ and $B$. Adding the wave functions yields a total wave function $\psi_T$ satisfying:

\[ \psi_T^\dagger \hat{\sigma} \psi_T = (\psi_A + \psi_B)^\dagger \hat{\sigma} (\psi_A + \psi_B) \]
\[ = \psi_A^\dagger \hat{\sigma} \psi_A + \psi_B^\dagger \hat{\sigma} \psi_B + \psi_A^\dagger \hat{\sigma} \psi_B + \psi_B^\dagger \hat{\sigma} \psi_A . \]  

(2)

Since the spins must be additive, the total wave function is not generally the sum of the individual wave functions. However, we can treat the wave functions as being independent if the interference terms cancel [1]. This cancelation imposes a vector constraint on the wave functions:

\[ \psi_A^\dagger \hat{\sigma} \psi_B + \psi_B^\dagger \hat{\sigma} \psi_A = 0 . \]  

(3)

Assuming either of the waves to be a spin eigenfunction everywhere, one component of this constraint requires the wave functions to anti-commute:

\[ \psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A = 0 . \]  

(4)

For waves representing identical particles, this is the Pauli exclusion principle. Hence we can conclude that standing waves described by spin eigenfunctions are fermions.

The anti-commutation of wave functions is not true in general, but we can force the cancellation by introducing a phase shift at each point between the two wave functions. Such phase shifts have no effect on the actual dynamics of the total wave, but allow us to pretend that each particle wave maintains its separate identity even though there is actually only one combined wave. Of course, this procedure is only valid if the particles interact weakly enough to remain distinguishable during the interaction. This limitation does not invalidate the basic premise that physical quantities are fully determined by the spin density field.

The phase shift ($\delta$) is determined from the constraint:

\[ \text{Re}(\psi_A^\dagger \exp(i\delta)\psi_B) = 0 , \]  
\[ \text{Re}(\psi_A^\dagger \psi_B) \cos \delta - \text{Im}(\psi_A^\dagger \psi_B) \sin \delta = 0 . \]  

(6)
This yields:

\[ \tan \delta = \frac{\text{Re}(\psi_A^\dagger \psi_B)}{\text{Im}(\psi_A^\dagger \psi_B)}. \]  

(7)

If we let \( \psi_A^\dagger \psi_B = g \exp i\beta \), then:

\[ \tan \delta = \cot \beta. \]

(8)

Therefore the phase angles are related by:

\[ \delta = \frac{\pi}{2} - \beta \pm n\pi \]

(9)

where \( n \) is an integer. Note that \( \delta \) is only unique within an arbitrary multiple of \( \pi \). The \( \pi/2 \) phase shift is a constant, so we will ignore it while analyzing phase shifts of individual particles.

Suppose we start with two wave functions \( \psi_A \) and \( \psi_B \), initially non-overlapping and normalized to one (\( \hbar \) will be multiplied explicitly to provide the dimension of angular momentum). We will assume that each particle is shifted by a phase that depends only on the other particle: \( \psi_A = \psi_A' \exp (i\varphi_B) \) and \( \psi_B = \psi_B' \exp (i\varphi_A) \), where the primed variables have zero interference. As they approach each other, the total wave function is \( \psi_T = \psi_A' + \psi_B' = \exp [-i\varphi_B] \psi_A + \exp [-i\varphi_A] \psi_B \).

The phase-shifted wave functions are independent, so they satisfy the free-particle wave equation, e.g.

\[ \hbar \partial_t (\exp [-i\varphi_B] \psi_A) = -iH_0 \exp [-i\varphi_B] \psi_A. \]

(10)

We take \( \psi_A \) to be an electron wave function with free particle hamiltonian \( H_0 \psi_A = (-c_1^5 \sigma \cdot i\hbar \nabla + m_e c^2 \gamma^0) \psi_A \).

Expanding the Dirac equation for \( \psi_A \) yields:

\[ (-i\hbar [\partial_t \varphi_B] + \hbar \partial_t) \psi_A = (-c_1^5 \sigma \cdot (-i\hbar [\nabla \varphi_B] + \nabla) - im_e c^2 \gamma^0) \psi_A \]

(11)

or

\[ (\hbar (i\partial_t + [\partial_t \varphi_B]) + \hbar c_1^5 \sigma \cdot (i\nabla + [\nabla \varphi_B]) - m_e c^2 \gamma^0) \psi_A = 0, \]

(12)

where square brackets indicate that the derivatives inside only apply to the immediately following variables.

The modified Hamiltonian is:

\[ H' \psi = i\hbar \partial_t \psi = -\hbar [\partial_t \varphi_B] \psi + c_1^5 \sigma \cdot \hbar (-i\nabla - [\nabla \varphi_B]) \psi + m_e c^2 \gamma^0 \psi. \]

(13)

This looks similar to the "minimal substitution" method for introducing electromagnetic potentials with \( q\Phi = -\hbar \partial_t \varphi_B \) and \( q\mathbf{A} = \hbar \nabla \varphi_B \). However, with those definitions (and the usual momentum operator \( \mathbf{P} = -i\nabla - q\mathbf{A} \)) the electric field would be determined from \( q\mathbf{E} = \hbar (\nabla \partial_t \varphi_B - \partial_t \nabla \varphi_B) \). If the space and time derivatives commute then we have no electric field in the linear theory. And if we keep the nonlinear terms that have hitherto been neglected, then the electric field would be a pure gradient since the time derivative of \( \mathbf{A} \) is cancelled out. However, "minimal substitution" is not a physical principle.

A more logical approach is to simply compute the evolution of the wave momentum \( -\psi^\dagger i\nabla \psi \). The convective time derivative \( (d_t = \partial_t + c_1^5 \sigma \cdot \nabla) \) yields:

\[ d_t \mathbf{P} = \text{Re} \left\{ \hbar \psi_A^\dagger (\partial_t + c_1^5 \sigma \cdot \nabla) (-i\nabla) \psi_A \right\}. \]

(14)

The independent phase-shifted wave function is unaffected:

\[ \hbar \left\{ \psi_A^\dagger \exp [i\varphi_B] (\partial_t + c_1^5 \sigma \cdot \nabla) (-i\nabla) \exp [-i\varphi_B] \psi_A \right\} + c.c. = 0. \]

(15)

The temporal derivatives yield:

\[ \hbar \left\{ \psi_A^\dagger \partial_t (-i\nabla) \psi_A - \psi_A^\dagger [\partial_t \varphi_B] \nabla \psi_A + \psi_A^\dagger [\partial_t (-i\nabla \varphi_B)] \psi_A + \psi_A^\dagger [\nabla \varphi_B \partial_t \varphi_B] \psi_A - \psi_A^\dagger [\nabla \varphi_B \partial_t \psi_A] \right\} + c.c. \]

(16)
To simplify analysis, we define the vector potential by \( qA \equiv -\hbar \nabla \varphi_B \), the charge density by \( \rho_A \equiv q\psi_A^\dagger \psi_A \), and the current by \( J_A \equiv \psi_A^\dagger q c \gamma^5 \sigma \psi_A \). The first term above represents the partial derivative \( \partial_t \mathbf{P} \). The fourth term is purely imaginary and cancelled by the complex conjugate. The other terms are:

\[
-\hbar \partial_t \varphi_B \nabla (\rho_A/q) + 2\rho_A \partial_t A + A \partial_t \rho_A.
\]

This is equivalent to:

\[
-\hbar \nabla (\rho_A \partial_t \varphi_B) + \rho_A \partial_t A + A \partial_t \rho_A.
\]

The spatial derivatives yield:

\[
\hbar \{ \psi_A^\dagger c \gamma^5 \sigma \cdot \nabla (-i\nabla) \psi_A - \psi_A^\dagger [c \gamma^5 \sigma \cdot \nabla \varphi_B] \nabla \psi_A + \psi_A^\dagger [c \gamma^5 \sigma \cdot \nabla (-\nabla \varphi_B)] \psi_A
+ \psi_A^\dagger [\nabla \varphi_B ic \gamma^5 \sigma \cdot \nabla \varphi_B] \psi_A - \psi_A^\dagger [\nabla \varphi_B] c\gamma^5 \sigma \cdot \nabla \psi_A \} + c.c.
\]

The first term is the convective part of \( d_t \mathbf{P} \). The fourth term is purely imaginary and cancelled by the complex conjugate. In terms of components, the three remaining terms represent:

\[
(\partial_t J_{A_j}) A_j + 2J_{A_j} \partial_t A_i + A_i \partial_j J_{A_j}.
\]

This is equivalent to:

\[
\partial_t (J_{A_J} A_J) + J_{A_J} (\partial_t A_i - \partial_i A_J) + \partial_j (A_i J_{A_J}).
\]

Combining terms and solving for \( d_t \mathbf{P} \) yields:

\[
d_t \mathbf{P} = \hbar \nabla (\rho A \partial_t \varphi_B) - \rho_A \partial_t A - A \partial_t \rho_A - \partial_t (J_{A_J} A_J) - J_{A_J} (\partial_t A_i - \partial_i A_J) - \partial_j (A_i J_{A_J}).
\]

When integrated over space, the total derivatives will yield zero, assuming that the wave fields fall to zero sufficiently rapidly at infinity (otherwise we need to consider a radiated photon). We also assume that phase varies much more rapidly than charge density: \( |\rho_A \partial_t A| \gg |A \partial_t \rho_A| \). This is closely related to the assumption of incident plane waves in quantum mechanical calculations. This leaves:

\[
d_t \mathbf{P} = -\rho_A \partial_t A + J_A \times (\nabla \times A).
\]

This is equivalent to the Lorentz force if:

\[
\mathbf{E} = -\partial_t \mathbf{A} (24a)
\]

\[
\mathbf{B} = \nabla \times \mathbf{A} (24b).
\]

This is the Weyl (or temporal) gauge. We can change the electric field to the usual form \( \mathbf{E} = -\nabla \Phi - \partial_t \mathbf{A} \) by performing a Helmholtz decomposition. Changing \( \mathbf{A} \) to \( \mathbf{A} \) does not affect the magnetic field calculation.

Others have similarly identified the vector potential \( \mathbf{A} \) as the gradient of a multivalued field.\[22–24\] The curl of the such gradients need not be identically zero. This interpretation is also consistent with Synge’s "primitive quantization" in which Planck’s constant \( \hbar \) represents the action for a single wave cycle.\[25\]

Suppose the field \( \psi_B \) produces a phase shift on \( \psi_A \) that varies by some multiple of \( \pi \) along a closed path:

\[
\oint \nabla \varphi_B \cdot d\ell = -n \pi
\]

for some integer \( n \). Stoke’s law yields quantization of magnetic flux:

\[
\iint \mathbf{B} \cdot d\mathbf{S} = \oint A \cdot d\ell = n\pi \frac{\hbar}{q} = n \frac{\hbar}{2q}.
\]

This classical quantization of magnetic flux is consistent with de Broglie’s observation in a 1963 interview that "... in quantum phenomena one obtains quantum numbers, which are rarely found in mechanics but occur very frequently in wave phenomena and in all problems dealing with wave motion."\[26\]
Alternatively, suppose that the interaction phase shift has the form:

\[ \phi_B = \frac{e^2}{4\pi \epsilon_0 \hbar \omega} (m \phi - \omega t) \left( \int \frac{|\psi_B(r', t)|^2}{|r - r'|} d^3r' \right) \tag{27} \]

with \( \hbar \omega \approx m_{e \gamma}^2 \) (corresponding to the frequency of the real-valued vector field, assuming that the spinor field has frequency \( m_{e \gamma}^2 / \hbar \)). The azimuthal dependence \((m \phi - \omega t)\) is what one might expect for a spherical harmonic wave. The radial dependence is purely speculative, since we don’t know the actual wave functions of isolated elementary particles.

The vector potential is then:

\[ A = -(\hbar/e) \nabla \phi_B = \frac{e}{4\pi \epsilon_0 \omega} \left( \frac{m \phi}{r \sin \theta} \right) \int \frac{|\psi_B(r', t)|^2}{|r - r'|} d^3r' + (m \phi - \omega t) \int \frac{|\psi_B(r', t)|^2 (r - r')}{|r - r'|^3} d^3r' \tag{28} \]

The electric field is that of a negative charge distribution:

\[ E = -\partial_t A = -\frac{e}{4\pi \epsilon_0} \int \frac{|\psi_B(r', t)|^2 (r - r')}{|r - r'|^3} d^3r' \tag{29a} \]

\[ \Phi = -\frac{e}{4\pi \epsilon_0} \int \frac{|\psi_B(r', t)|^2}{|r - r'|} d^3r'. \tag{29b} \]

The magnetic flux inside a circle of radius \( r \) centered at \( r' = 0 \) in the plane \( z = 0 \) is:

\[ \oint A \cdot d\ell = \frac{e}{4\pi \epsilon_0 \omega} \left( 2 \frac{m \phi}{\sin \theta} \int \frac{|\psi_B(r', t)|^2}{|r - r'|} d^3r' \right). \tag{30} \]

For large \( r \) we use the fact that \( |\psi_B|^2 \) is normalized to one and make the approximation:

\[ \int \frac{|\psi_B(r', t)|^2}{|r - r'|} d^3r' \approx \frac{1}{r}. \tag{31} \]

This amounts to treating the electron as a point-like particle. The magnetic flux is then:

\[ \oint A \cdot d\ell = \frac{m \phi \mu_0 \hbar e}{2 \epsilon_0 \omega r} = \frac{m \phi \mu_0 \hbar e}{4m_{e \gamma} r}. \tag{32} \]

For \( m_\phi = 1 \), this is \( \mu_0 \hbar e / (4m_{e \gamma}) \). For comparison, the electron spin dipole moment is \( M \approx \hbar c / (2m_e) \) to first order with magnetic field:

\[ B = \frac{\mu_0 M}{4\pi r^2} \left( 2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta} \right). \tag{33} \]

In the \( z = 0 \) plane, the field is entirely in the \( z \)-direction:

\[ B_z(z = 0) = -\frac{\mu_0 M}{4\pi r}. \tag{34} \]

and its magnetic flux in the plane \( z = 0 \) is:

\[ \oint B \cdot \hat{\mathbf{n}} dS = \int -2\pi \frac{\mu_0 M}{4\pi r} r dr = \frac{\mu_0 M}{2r} = \mu_0 \hbar e / (4m_{e \gamma}), \tag{35} \]

in agreement with our calculation. Hence this choice of phase shift, consistent with interpretation of the real-valued wave function as a vector spherical harmonic, yields the correct relationship between the electron’s electric field and magnetic flux.
2.0.2. Maxwell’s Equations

The electromagnetic fields defined above are also subject to Maxwell’s equations. The definitions of \( \mathbf{E} \) and \( \mathbf{B} \) imply Faraday’s Law and Gauss’ magnetic law:

\[
\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \tag{36a}
\]

Gauss’ electric law and Ampere’s law define the charge and current densities (\( \rho_c \) and \( \mathbf{J} \), respectively):

\[
\nabla \cdot \mathbf{E} = -(\nabla \cdot \partial_t \mathbf{A} + \nabla^2 \Phi) \equiv \frac{\rho_c}{\epsilon_0}, \quad \nabla \times \mathbf{B} - \frac{1}{c^2} \partial_t \mathbf{E} = \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} (\partial_t^2 \mathbf{A} + \partial_t \nabla \Phi) \equiv \mu_0 \mathbf{J}. \tag{37b}
\]

These definitions of charge and current densities are consistent with the continuity equation:

\[
\partial_t \rho_c + \nabla \cdot \mathbf{J} = 0. \tag{38}
\]

Hence particle-like waves in an elastic solid can behave like fermions, with electromagnetic potentials derived from phase shifts that result from wave interference.

2.0.3. Quantum Electrodynamics

It is customary in quantum mechanics textbooks to define \( \bar{\psi} \equiv \psi^\dagger \gamma^0 \), replace \( \psi^\dagger \) with \( \bar{\psi} \gamma^0 \), and define the ”4-vector” of matrices \( \gamma^\mu \equiv (\gamma^0, \gamma^0 \gamma^5 \sigma) \). The 4-potential is \( \mathbf{A}^\mu = (\Phi, -\mathbf{A}) \) and the 4-current (\( \rho, \mathbf{J} \)) is \( \mathbf{J}^\mu = q \bar{\psi} \gamma^\mu \psi \). These changes of variables are intended to make the theory look more ”relativistic”. It is also common to use ”natural” units with \( \mu_0 = \epsilon_0 = c = 1 \). Using this notation with \( \partial^\mu = (\partial_t, \nabla) \), the Lagrangian density for two interacting electrons is:

\[
\mathcal{L} = \bar{\psi}_A [\gamma^\mu (i \partial_\mu - q A_\mu) - m_A] \psi_A + \bar{\psi}_B [\gamma^\mu (i \partial_\mu - q A_\mu) - m_B] \psi_B. \tag{39}
\]

Separating the interaction of particle \( B \) yields:

\[
\mathcal{L} = \bar{\psi}_A [\gamma^\mu (i \partial_\mu - q A_\mu) - m_A] \psi_A + \bar{\psi}_B [\gamma^\mu (i \partial_\mu - q A_\mu) - m_B] \psi_B - J^\mu A_\mu. \tag{40}
\]

Since the Dirac equation is satisfied for each particle, this is equivalent to:

\[
\mathcal{L} = \bar{\psi}_A [\gamma^\mu (i \partial_\mu - q A_\mu) - m_A] \psi_A + J^\mu A_\mu - J^\mu A_\mu. \tag{41}
\]

Relationships between potentials and sources are given in Eqs. 37. Assuming time-independence with zero divergence of the vector potential and zero curl of the electric field, the sources become:

\[
\frac{\rho_c}{\epsilon_0} = -\nabla^2 \Phi, \quad \mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}). \tag{42a}
\]

Therefore:

\[
J^\mu A_\mu = -\Phi \nabla^2 \Phi - \mathbf{A} \cdot (\nabla \times \nabla \times \mathbf{A}). \tag{43}
\]

According to Green’s first identity:

\[
- \int_V \Phi \nabla^2 \Phi dV = \int_V (\nabla \Phi)^2 dV - \int_{\partial V} \Phi \mathbf{n} \cdot \nabla \Phi dS. \tag{44}
\]

Similarly:

\[
- \int_V \mathbf{A} \cdot (\nabla \times \nabla \times \mathbf{A}) dV = - \int_V (\nabla \times \mathbf{A})^2 dV + \int_{\partial V} \mathbf{A} \times (\nabla \times \mathbf{A}) dS. \tag{45}
\]
Using the definitions of \( E \) and \( B \) while neglecting boundary integrals yields:
\[
\mathcal{L} = \bar{\psi}_A \left[ \gamma^\mu \left( i \partial_\mu - q A_\mu \right) - m_A \right] \psi_A + \left( E^2 - B^2 \right) - J^\mu A_\mu
\]

This differs from the (non-quantized) Lagrangian density of quantum electrodynamics (QED) by a factor of 1/2 in front of \( (E^2 - B^2) \). This difference is resolved by the fact that when varying the potentials \( A_\mu \), the source densities \( J^\mu \) should be regarded as functions of \( A_\mu \). However, it is conventional to vary the potentials independently of the source densities, yielding only half of the correct value. When computing variations of \( E^2 \) and \( B^2 \), both factors in \( E^2 \) (and \( B^2 \)) are varied. To eliminate this double-counting and be consistent with independent variation of the potentials, a factor of 1/2 must be introduced:
\[
\mathcal{L} = \bar{\psi}_A \left[ \gamma^\mu \left( i \partial_\mu - q A_\mu \right) - m_A \right] \psi_A + \frac{1}{2} \left( E^2 - B^2 \right) - J^\mu A_\mu
\]

This is the Lagrangian density of non-quantized QED, in which a single charged fermion interacts with an electromagnetic field. Generalization to multiple particles requires a quantization procedure.

### 3. DISCUSSION

We have outlined a similarity between a classical model of interacting waves in an elastic solid to quantum electrodynamics (QED). This interpretation of QED, and by extension the Standard Model, is that it represents a decomposition of the classical spin density field into interacting elementary particles. Others have also associated quantum mechanical behavior with waves in an elastic solid. [10, 27–30]

### 4. CONCLUSIONS

This paper describes interactions of classical waves of spin density. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials, with suggested interpretations of magnetic flux quantization and the Coulomb potential. The Lagrangian density of single-fermion quantum electrodynamics is also given a classical physics interpretation. Hence classical wave theory offers insight into the physical basis for relativistic quantum mechanics.