An Introduction to Wave Mechanics

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This paper offers a conceptually simple introduction to wave mechanics, the description of elementary particles as consisting of waves. The paper specifically addresses two common misconceptions by demonstrating that special relativity and spin angular momentum are consequences of classical wave theory. First, a wave equation is derived for infinitesimal shear waves in an elastic solid. Next, a change of variables is used to describe the waves in terms of classical spin angular momentum density, which is the field whose curl is equal to twice the classical momentum density. The second-order wave equation is then converted to a first-order Dirac equation. Conceptually, the Dirac equation is much easier to understand than the Schrödinger equation for two reasons. First, each term has a well-defined dynamical interpretation, and second, consistency with special relativity is guaranteed by Lorentz-invariance of the wave equation. Plane wave solutions are presented, and the dynamical operators of relativistic quantum mechanics are derived. Wave interference gives rise to the Pauli exclusion principle and electromagnetic potentials.
I. INTRODUCTION

Students of physics are typically introduced to quantum mechanics via the Schrödinger equation. Although this equation is often considered to be the first wave-like description of matter, it suffers from the fact that unlike ordinary wave equations, it is not Lorentz-invariant. The Schrödinger equation also does not provide an interpretation of spin angular momentum, which is intrinsic to elementary particles. Isaac Newton reportedly said, “A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding.” Although the Dirac equation may seem more complicated than the Schrödinger equation, it has the advantage of being a physically realistic, and therefore comprehensible, description of nature. Therefore we propose that the Dirac equation, which is both relativistic and describes spin angular momentum, is a better starting point for understanding quantum mechanics.

The Dirac formalism has been used by various researchers to describe classical wave dynamics. [1–8] In particular, analysis of shear waves in an ideal elastic solid yields a Dirac equation with the same dynamical operators as relativistic quantum mechanics. [1–3] That is the model used in this paper.

In Section II, we derive a Lagrangian and second-order wave equation for infinitesimal shear waves in an elastic solid. In Section III, we explain the physical interpretation of spin angular momentum. In Section IV, we derive quantum mechanics. [1–3] That is the model used in this paper.

In Section V, we describe how wave interactions give rise to the Pauli exclusion principle and electromagnetic potentials.

II. IDEAL ELASTIC SOLID

We consider the case of an isotropic, homogeneous solid with a linear relationship between infinitesimal stress and strain. The usual expression for potential energy is:

\[
\int U \, d^3r = \int \left( \frac{1}{2} \lambda (\nabla \cdot \xi)^2 + \mu e_{ij}^2 \right) d^3r
\]  

(1)

where \(e_{ij}\) is the symmetric strain tensor \((\partial_i \xi_j + \partial_j \xi_i)/2\), and \(\lambda\) and \(\mu\) are the Lame’ parameters. This expression has the drawback that it does not cleanly separate compressible and rotational motion. We can remedy this as follows:

Expanding the square of the symmetrical strain tensor yields:

\[
e_{ij}^2 = (\partial_i \xi_j)^2 + (\partial_j \xi_i)^2 + (\partial_i \xi_i)^2
\]

\[
+ \frac{1}{2} ((\partial_j \xi_j + \partial_j \xi_j)^2 + (\partial_j \xi_j + \partial_j \xi_j)^2 + (\partial_j \xi_j + \partial_j \xi_j)^2).
\]

(2)

Add \(2(\partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j)\) to the first term and subtract it from the second term to obtain:

\[
e_{ij}^2 = (\nabla \cdot \xi)^2
\]

\[
+ \frac{1}{2} ((\partial_j \xi_j + \partial_j \xi_j)^2 + (\partial_j \xi_j + \partial_j \xi_j)^2 + (\partial_j \xi_j + \partial_j \xi_j)^2)
\]

\[
- 2(\partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j).
\]

(3)

Integrate the extra terms by parts on each of the two derivatives (assuming no contribution at infinity) to obtain:

\[
e_{ij}^2 \rightarrow (\nabla \cdot \xi)^2
\]

\[
+ \frac{1}{2} ((\partial_j \xi_j + \partial_j \xi_j)^2 + (\partial_j \xi_j + \partial_j \xi_j)^2 + (\partial_j \xi_j + \partial_j \xi_j)^2)
\]

\[
- 2(\partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j + \partial_j \xi_j).
\]

(4)

This is equivalent to:

\[
e_{ij}^2 \rightarrow (\nabla \cdot \xi)^2 + \frac{1}{2}(\nabla \times \xi)^2.
\]

(5)

The potential energy density may therefore be expressed as:

\[
U = \frac{1}{2} (\lambda + 2\mu)(\nabla \cdot \xi)^2 + \frac{1}{2}\mu(\nabla \times \xi)^2.
\]

(6)
This form of the potential energy density separates infinitesimal irrotational and incompressible motion. It is a quadratic function of the first derivatives of displacement. The Lagrangian for infinitesimal incompressible motion is:

\[ L = \int \left( \frac{1}{2} \rho (\partial_t \xi)^2 - \frac{1}{2} \mu (\nabla \times \xi)^2 \right) dV. \] (7)

The Euler-Lagrange equation is the usual equation for infinitesimal shear waves:

\[ \partial_t^2 \xi = -\frac{\mu}{\rho} \nabla \times \nabla \times \xi \] (8)

for which the wave speed is \( c = \sqrt{\mu/\rho} \).

The incompressible potential energy in Eq. 7 is the form of potential energy used by MacCullagh in 1837 to describe light waves [9]. In spite of this well-known result, some have claimed that the rotation vector \((\nabla \times \xi)/2\) cannot appear in the energy density because the energy should be independent of rotations (see e.g. Ref. 10). That logic does not apply here for two reasons. First, we assume zero displacement at infinity, so global rotations are not allowed. Second, elastic energy density depends on relative motion at different positions, so integration by parts can alter the spatial distribution of energy density. There is no need for so-called "second-gradient" elasticity (inclusion of derivatives of rotation in the energy density), since the energy density depends directly on the infinitesimal rotation vector.

### III. SPIN ANGULAR MOMENTUM

It is well known that elastic waves in solids have two types of momentum: that of the medium \((\rho \partial_t \xi)\) and that of the wave: \(\rho(\nabla \xi_j)\partial_t \xi_j\) (see e.g. Ref. 11). Clearly there must also be two types of angular momentum in an elastic solid: "spin" associated with rotation of the medium, and "orbital" associated with rotation of the wave. However, spin angular momentum is not normally considered to be a classical physics concept.

Unlike the "moment of momentum" definition \(r \times p\), spin angular momentum density is an intrinsic property defined at each point in space. Coordinate-independent descriptions of rotational dynamics can actually be traced back to the nineteenth century [12]. In 1891 Oliver Heaviside recognized MacCullagh’s force density in Eq. 8 as being the curl of a torque density that is proportional to an infinitesimal rotation angle \(\Theta = (1/2)\nabla \times \xi\). [13] However, this idea seems to have been largely forgotten.

The key to understanding classical spin angular momentum is the Helmholtz decomposition of momentum density. The momentum density \(p = \rho v\) consists of an incompressible (or rotational) part (\(\tilde{p}\)), an irrotational (or compressible) part (\(\hat{p}\)), and a constant part (\(\bar{p}\)) determined by the Helmholtz decomposition:

\[ p = \tilde{p} + \hat{p} + \bar{p} = \frac{1}{2} \nabla \times s - \nabla \Phi + \bar{p} \] (9)

where:

\[ s(r, t) = \frac{1}{2\pi} \nabla \times \int_V \frac{p(r', t) - \hat{p}}{|r - r'|} dV' \] (10)

\[ \Phi(r, t) = -\frac{1}{4\pi} \nabla \cdot \int_V \frac{p(r', t) - \hat{p}}{|r - r'|} dV'. \] (11)

Previous work has demonstrated that \(s\) represents angular momentum density corresponding to spin in relativistic quantum mechanics. [1–3, 14] Hence we refer to \(s\) as "spin density".

Assuming sufficiently rapid fall-off at large distances, the volume integral of spin density is equal to the volume integral of \(r \times \hat{p}\). The two representations of angular momentum density are related by integration by parts [3]:

\[ \int r \times \frac{1}{2} (\nabla \times s) d^3r = \frac{1}{2} \int (\nabla (r \cdot s) - r \cdot \nabla s - s \cdot \nabla r) d^3r \]

\[ = \frac{1}{2} \int (\nabla (r \cdot s) - \partial_i (r_i s) + s(\nabla \cdot r) - s \cdot \nabla r) d^3r \]

\[ = \int s d^3r \] (12)

where the total derivatives are assumed not to contribute to the last line, since they can be converted into surface integrals that are assumed to vanish.
The rotational kinetic energy is similarly [3]:

\[
K = \frac{1}{2\rho} \int \rho \hat{r}^2 d^3r = \frac{1}{2\rho} \int \left[ \frac{1}{2} \nabla \times \mathbf{s} \right]^2 dV \\
= \frac{1}{8\rho} \int \left[ \nabla \cdot (\mathbf{s} \times (\nabla \times \mathbf{s})) + \mathbf{s} \cdot [\nabla \times (\nabla \times \mathbf{s})] \right] dV \\
= \frac{1}{2} \int \mathbf{w} \cdot \mathbf{s} dV.
\]

(13)

where \(\mathbf{w} = \nabla \times \mathbf{v}/2\) is the angular velocity (sometimes confusingly referred to as "spin" in the literature). In this case the divergence term is assumed not to contribute to the integral.

The second line in Eq. 13 may be rewritten as:

\[
K = \frac{1}{2} \int_V \dot{\Theta} \cdot \mathbf{s} dV + \frac{1}{4} \int_{\partial V} \hat{n} \cdot (\mathbf{s} \times \mathbf{v}) dA.
\]

(14)

If the boundary integral can be neglected, then \(\mathbf{s}\) is the momentum conjugate to angular velocity:

\[
\frac{\delta}{\delta \Theta_i} \int \frac{1}{2} \dot{\Theta}_j \dot{s}_j dV = \frac{1}{2} s_i + \int \frac{1}{2} \dot{\Theta}_j \frac{\delta}{\delta \Theta_i} s_j dV = \frac{1}{2} s_i + \int \frac{1}{2} s_j \frac{\delta}{\delta \Theta_i} \dot{\Theta}_j dV = s_i
\]

(15)

where integration by parts was used twice to evaluate the final integral.

As an example, consider a cylinder of radius \(R\) aligned with the \(z\)-axis and rotating rigidly with angular velocity \(w_0\). The motion is described by these non-zero variables:

\[
S_z = \rho w_0[R^2 - r^2] \quad \text{for} \quad r \leq R \quad \text{and zero for} \quad r > R;
\]

(16)

\[
v_\phi = -\frac{1}{2} \frac{\partial}{\partial r} S_z = r w_0 \quad \text{for} \quad r \leq R \quad \text{and zero for} \quad r > R;
\]

(17)

\[
w_z = \frac{1}{2r} \frac{\partial}{\partial r} r v_\phi = w_0 \left[1 - R \delta(r - R)/2\right] \quad \text{for} \quad r \leq R \quad \text{and zero for} \quad r > R.
\]

(18)

The total angular momentum per unit height is

\[
J = 2\pi \int_0^R S_z r dr = 2\pi \int_0^R \rho w_0[R^2 - r^2] r dr \\
= \frac{1}{2} \pi \rho R^4 w_0 = \frac{1}{2} MR^2 w_0 \\
= I w_0.
\]

(19)

where we have used the mass per unit height \(M = \rho \pi R^2\) and moment of inertia per unit height \(I = MR^2/2\).

The kinetic energy per unit height is

\[
K = \frac{1}{2} \int \mathbf{w} \cdot \mathbf{s} r dr d\phi = \pi \int_0^R w_0 \left[1 - R \delta(r - R)/2\right] \rho w_0[R^2 - r^2] r dr \\
= \pi \rho w_0^2 \left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{MR^2 w_0^2}{4} = \frac{1}{2} I w_0^2.
\]

(20)

These are in agreement with standard rotational dynamics.

Defining \(\Theta = (1/2)\nabla \times \xi\), Eq. 8 becomes:

\[
\partial_t (\nabla \times \mathbf{s}) + 4\mu \nabla \times \Theta = 0.
\]

(21)

The reader should note that \(\Theta\) is a vector and only represents an angle for infinitesimal motion. Assuming \(\nabla \cdot \mathbf{s} = 0\), the Helmholtz decomposition yields:

\[
\partial_t \mathbf{s} + 4\mu \Theta = 0.
\]

(22)

This equation states that the rate of change of angular momentum density is equal to torque density, which is proportional to infinitesimal rotation angle.
The next step is to relate the displacement $\xi$ to the spin density $s$. For infinitesimal motion, define a vector potential $Q$ such that $\partial_t Q = s$. Since the curl of $s$ is proportional to velocity, the curl of $Q$ must be proportional to displacement:

$$\frac{1}{2\rho} \nabla \times Q = \xi$$

Therefore the equation for $s$ is equivalent to:

$$\partial_t^2 Q + c^2 \nabla \times \nabla \times Q = 0.$$  \hspace{1cm} (24)

where $c^2 = \mu/\rho$. The curl of this equation yields Eq. 8.

Thus far we have assumed infinitesimal motion. Previous work attempted to describe finite motion by the equation:

$$\partial_t s + v \cdot \nabla s - w \times s = -c^2 \nabla \times \nabla \times Q = \tau.$$  \hspace{1cm} (25)

The logic of this equation is that changes due to translation ($v \cdot \nabla s$) and rotation ($-w \times s$) do not require torque density ($\tau$). Similarly, the momentum density equation may be interpreted as a statement that changes due to translation ($v \cdot \nabla p$) do not require force density ($f$):

$$\partial_t p + v \cdot \nabla p = f.$$  \hspace{1cm} (26)

However, it is not clear how to reconcile these two equations. In this paper we only consider infinitesimal motion.

IV. DIRAC EQUATION

The wave equation is a second order vector equation. In order to use variational methods, it is desirable to re-write the wave equation as a first order equation. In order to do this, we will follow Ref. [3] by starting with one-dimensional waves and then generalizing to three dimensions.

A. One-Dimensional Waves

Consider a one-dimensional wave propagating in one-dimension with amplitude of $a(z,t)$. If the wave equation is

$$\partial^2_t a = c^2 \partial^2_z a,$$  \hspace{1cm} (27)

then the general solution consists of backward ($B$) and forward ($F$) propagating waves:

$$a = a_B(ct + z) + a_F(ct - z)$$  \hspace{1cm} (28)

The two directions of wave propagation are clearly independent states, and they are separated in space by a 180° rotation. This property is the fundamental characteristic of spin one-half functions. Generalization to three dimensional space should therefore involve Dirac wave functions.

The forward and backward waves satisfy the equations:

$$\partial_t a_B = \partial_z a_B$$
$$\partial_t a_F = -\partial_z a_F.$$  \hspace{1cm} (29)

Defining $\dot{a} = \partial_t a$, we can write the wave equation as a first-order matrix equation:

$$\partial_t \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \partial_z \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} = 0.$$  \hspace{1cm} (30)

We have thus achieved our goal of converting a one-dimensional second-order wave equation into a first-order matrix equation. Generalization to three dimensional vector waves requires additional components. One possibility is to introduce vector components such as $(a_{B_i})$ and $(a_{F_i})$ to make a 6-element column vector in the equation above. Unfortunately, this method does not allow a simple means for changing the direction of the derivative ($\partial_z$). Therefore we follow a different path.
First, note that the procedure above specifies independent components with positive and negative wave velocity, and uses a diagonal matrix to relate spatial and temporal derivatives. We can apply a similar technique to separate positive and negative values of the wave function. Letting \( a_B \) and \( a_F \) represent the \( z \)-components of vectors, separate each component of the wave into positive and negative parts (\( \hat{a}_B = \hat{a}_{B+} - \hat{a}_{B-} \) and \( \hat{a}_F = \hat{a}_{F+} - \hat{a}_{F-} \)) so that each of the four wave components (\( \hat{a}_{B+}, \hat{a}_{B-}, \hat{a}_{F+}, \hat{a}_{F-} \)) is positive-definite. With these definitions, we have:

\[
a_B' = \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{B-}^{1/2} \end{bmatrix} = \eta^T \sigma_z \eta,\tag{31}
\]

and similarly for \( \dot{a}_F \). The matrix \( \sigma_z \) is the Pauli spin matrix for the \( z \)-component of a vector. Notice that this matrix has the same form (within a minus sign) as the velocity direction matrix in Eq. 30. In one dimension, the significance of simultaneous positive and negative components is unclear. We will see that in three dimensions, simultaneous positive and negative components for one direction indicates polarization in a different direction.

We arrange the components in the following order, corresponding to the chiral representation of the Dirac wave function:

\[
\begin{bmatrix} \hat{a}_{B+} \\ \hat{a}_{F-} \\ \hat{a}_{F+} \\ \hat{a}_{B-} \end{bmatrix}.	ag{32}
\]

We may now write the time derivative \( \dot{a} \) as the matrix product:

\[
\dot{a} = \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{B-}^{1/2} \\ \hat{a}_{F+}^{1/2} \\ \hat{a}_{F-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{B-}^{1/2} \\ \hat{a}_{F+}^{1/2} \\ \hat{a}_{F-}^{1/2} \end{bmatrix} = \psi^T \sigma_z \psi \tag{33}
\]

where \( \sigma_z \) now represents the \( 4 \times 4 \) Dirac matrix for the \( z \)-component of spin density, and the the four-component column vectors are called Dirac bispinors. The spatial derivative is now given by:

\[
c \partial_z a = - \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{F-}^{1/2} \\ \hat{a}_{F+}^{1/2} \\ \hat{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{F-}^{1/2} \\ \hat{a}_{F+}^{1/2} \\ \hat{a}_{B-}^{1/2} \end{bmatrix} = -\psi^T \gamma^5 \psi \tag{34}
\]

where an overall minus sign has been introduced in order to maintain consistency with the chiral representation of the Dirac equation. The matrix \( \gamma^5 \) is the Dirac matrix for chirality. If the amplitude \( \langle a \rangle \) represents rotation angle, then positive and negative chirality \( \langle \partial_z a \rangle \) are analogous to left- and right-handed threads on a screw. Wave velocity \( \langle v \rangle \) is obtained by combining the two matrices used above:

\[
\psi = c \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \hat{a}_{B+}^{1/2} \\ \hat{a}_{F-}^{1/2} \\ \hat{a}_{F+}^{1/2} \\ \hat{a}_{B-}^{1/2} \end{bmatrix} = c \gamma^5 \psi \tag{35}
\]

The one-dimensional scalar wave equation may be written in the form:

\[
\partial_t [\psi^T \sigma_z \psi] + c \partial_z [\psi^T \gamma^5 \psi] = \partial_t^2 a - c^2 \partial_z^2 a = 0.\tag{36}
\]

Other matrices may be inserted between the wave functions, resulting in the following corresponding expressions (correcting a mistake in Ref. [3]). Each of these is equal to zero for the wave solutions:

\[
\partial_t [\psi^T \psi] + c \partial_z [\psi^T \gamma^5 \psi] = \partial_t [\partial_t a_F] + \partial_t [\partial_t a_B] + c^2 (\partial_z [\partial_z a_F] - \partial_z [\partial_z a_B]); \tag{37}
\]

\[
\partial_t [\psi^T \gamma^5 \psi] + c \partial_z [\psi^T \psi] = c (\partial_t [\partial_z a_F] - \partial_z [\partial_z a_B] + \partial_z [\partial_t a_F] + \partial_t [\partial_t a_B]); \tag{38}
\]

\[
\partial_t [\psi^T \gamma^5 \psi] + c \partial_z [\psi^T \sigma_z \psi] = \partial_t (-c \partial_z a) + c \partial_z [\partial_t a]. \tag{39}
\]
B. Three-Dimensional Vector Waves

Generalization to three dimensions is based on the fact that the matrix $\sigma_z$ may be regarded as representing one component of a three-dimensional vector. If we allow the 2-element column vector $\eta$ to have complex components, then it is a Pauli spinor, and the three components of a vector are:

$$a_x = \eta^\dagger \sigma_x \eta = \eta^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eta$$
$$a_y = \eta^\dagger \sigma_y \eta = \eta^\dagger \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \eta$$
$$a_z = \eta^\dagger \sigma_z \eta = \eta^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \eta.\quad(40)$$

The normalized spinor eigenfunctions for each direction are:

$$\sigma_x \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \sigma_y \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad \sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.\quad(41)$$

The algebra of the Pauli matrices is called "geometric algebra":

$$\sigma_x \sigma_y \sigma_z = iI, \quad \sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk} \sigma_k.\quad(42)$$

where the unit imaginary $"i"$ represents a unit oriented "volume". It is a true scalar if the matrices represent axial vectors, and is a pseudoscalar if the matrices represent polar vectors. The fourth independent matrix in this algebra is the identity matrix ($I$). For each propagation direction, the direction of the vector $\eta^\dagger \sigma \eta$ can be rotated by operations of the form:

$$R_{\phi_j}(\sigma_i) = \exp (-i\sigma_j \phi_j/2) \sigma_i \exp (i\sigma_j \phi_j/2) = (\cos (\phi_j/2) - i\sigma_j \sin (\phi_j/2)) \sigma_i (\cos (\phi_j/2) + i\sigma_j \sin (\phi_j/2))\quad(43)$$

These rotation matrices may operate either on the wave functions (Schrödinger picture) or on the matrices (Heisenberg picture).

Just as there are three Pauli matrices indicating different vector directions, there are also three orthogonal matrices associated with wave velocity. These are:

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^4 = -\begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}, \quad \gamma^5 = -\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\quad(44)$$

where $I_2$ is the $2 \times 2$ identity matrix. These matrices have the same form as the Pauli spin matrices except that two of the matrices have minus signs, as in a 180-degree rotation about the direction represented by $\gamma^0$. The unit imaginary here is denoted as $\tilde{i}$ because velocity is a polar vector, so this unit imaginary $\tilde{i} = \gamma^0 \gamma^4 \gamma^5$ is a pseudoscalar (spatial inversion changes $\tilde{i}$ to $-\tilde{i}$). [15]

The one-dimensional wave equation (Eq. 36) has the bispinor form:

$$\psi^T \{ \sigma_z \partial_t \psi + c\gamma^5 \partial_z \psi \} + \text{Transpose} = 0.\quad(45)$$

We can separate a common factor of $\psi^\dagger \sigma_z$:

$$\psi^\dagger \sigma_z \{ \partial_t \psi + c\gamma^5 \sigma_z \partial_z \psi \} + \text{Transpose} = 0.\quad(46)$$

For arbitrary vector components and derivatives, the bispinors are complex and the wave equation is:

$$\psi^\dagger \sigma_i \{ \partial_t \psi + c\gamma^5 \sigma_j \partial_j \psi \} + \text{adjoint} = 0.\quad(47)$$
This is the first-order wave equation for vector waves in three dimensions.

Expanding the spatial derivative term in Eq. (47) yields the 3-D generalization of the wave equation (36):

\[ 0 = \partial_t [\psi \gamma^0 \sigma^0 \psi] + c \nabla [\psi \gamma^5 \psi] - ic \left\{ \left[ \nabla \psi^\dagger \right] \times \gamma^5 \sigma \psi + \psi^\dagger \gamma^5 \sigma \times \nabla \psi \right\} \]

\[ = \partial_t^2 a - c^2 \nabla (\nabla \cdot a) + c^2 \nabla \times (\nabla \times a), \]

(48)

where corresponding terms are in the same order in each line of the equation. This is the result we have been seeking. We have rewritten the second-order vector wave equation as a first order equation involving Dirac bispinors. The validity of this correspondence, which we will confirm with examples, demonstrates that the Dirac equation of relativistic quantum mechanics is simply a special case of an ordinary vector wave equation.

Replacing the vector \( a \) by \( 2Q \) yields the following physical correspondences:

\[ s = \partial_t Q \equiv \frac{1}{2} [\psi^\dagger \sigma^0 \psi]; \]

\[ c \nabla \cdot Q \equiv -\frac{1}{2} [\psi^\dagger \gamma^5 \psi]; \]

\[ c^2 \{ \nabla \times \nabla \times Q \} = \frac{ic}{2} \left\{ \left[ \nabla \psi^\dagger \right] \times \gamma^5 \sigma \psi + \psi^\dagger \gamma^5 \sigma \times \nabla \psi \right\}; \]

\[ 0 = \frac{ic}{2} \nabla \cdot \left\{ \left[ \nabla \psi^\dagger \right] \times \gamma^5 \sigma \psi + \psi^\dagger \gamma^5 \sigma \times \nabla \psi \right\}. \]

(50)

(51)

(52)

These identifications provide seven independent constraints on the eight free parameters of the complex Dirac bispinor: three for the first, one for the second, two for the third (since a curl has only two independent components), and one for the fourth. There is also an arbitrary overall phase factor. The last identification simply states that the divergence of a curl is zero. This condition is necessary for consistency.

The first-order wave equation (Eq. 47) can be reduced to:

\[ \partial_t \psi + c \gamma^5 \sigma \cdot \nabla \psi + i \chi \psi = 0, \]

(53)

where \( \chi \) is any operator with the property

\[ \text{Re} \left\{ \psi^\dagger \sigma_j i \chi \psi \right\} = 0. \]

(54)

Behavior of the unit imaginary (and \( \chi \)) under spatial inversion is not specified and will be left undetermined from here on. The equation for an electron is obtained by the choosing \( \chi = M \gamma^0 \). Hence the Dirac equation for an electron may be interpreted as an ordinary wave equation with a clear dynamical interpretation describing the motion of an elastic solid.

Multiplying Eq. 53 by \( \psi^\dagger \) and adding the adjoint yields a conservation law with density \( \psi^\dagger \psi \) and current \( \psi^\dagger c \gamma^5 \sigma \psi \):

\[ \partial_t (\psi^\dagger \psi) + \nabla \cdot (\psi^\dagger c \gamma^5 \sigma \psi) = 0. \]

(55)

In quantum mechanics this equation is regarded as a conservation law for probability density, but in both classical and quantum mechanics it is part of the description of the evolution of spin angular momentum density.

C. Sample Plane Wave Solutions

As a simple mathematical example, the longitudinal wave \((Q_x, Q_y, Q_z) = (0, 0, Q_0 \sin (\omega t - kz))\) propagating along the z-axis may be expressed in the bispinor form:

\[ \psi = \sqrt{2\omega Q_0 \exp [-i (\omega t - kz)/2]} \begin{bmatrix} 0 \\ \sin (|\omega t - k|)/2 \\ \cos (|\omega t - k|)/2 \\ 0 \end{bmatrix} \]

(56)

The phase factor in front is necessary for consistency between the Dirac equation and the wave equation. For \( \omega = ck \), this wave function yields:

\[ s = \partial_t Q = \frac{1}{2} [\psi^\dagger \sigma^0 \psi] = (0, 0, \omega Q_0 \cos (\omega t - k z)); \]

\[ c \nabla \cdot Q = -\frac{1}{2} [\psi^\dagger \gamma^5 \psi] = -\omega Q_0 \cos (\omega t - k z); \]

\[ c^2 \left\{ \nabla \times \nabla \times Q \right\} = -\frac{ic}{2} \left\{ \left[ \nabla \psi^\dagger \right] \times \gamma^5 \sigma \psi + \psi^\dagger \gamma^5 \sigma \times \nabla \psi \right\} = (0, 0, 0); \]

(57a)

(57b)

(57c)
In this case \( \nabla \times \mathbf{s} = 0 \), so this wave solution is not relevant for describing shear waves in an elastic solid.

In addition to the wave variables described above, there are other "observables" that may be computed from the wave function. These include the vectors:

\[
\begin{align*}
\mathbf{b}_0 &= \psi^\dagger \gamma^0 \mathbf{\sigma} \psi = \mathbf{\hat{y}} \times \omega \mathbf{Q} = (\omega Q_0 \sin(\omega t - k z), 0, 0); \\
\mathbf{b}_4 &= \psi^\dagger \gamma^4 \mathbf{\sigma} \psi = \mathbf{\hat{x}} \times \omega \mathbf{Q} = (0, -\omega Q_0 \sin(\omega t - k z), 0); \\
\mathbf{b}_5 &= \psi^\dagger \gamma^5 \mathbf{\sigma} \psi = \omega Q_0 \mathbf{\hat{z}} = (0, 0, \omega Q_0);
\end{align*}
\]

These include the vectors: \( \gamma \) direction. This can be accomplished using the \( \mathbf{b} \)-vectors are orthogonal to \( \mathbf{b}_5 \) and to each other. Hence the matrices \( (\gamma^0, \gamma^4, \gamma^5) \) may be interpreted as defining directions relative to the wave velocity direction. The matrix \( \gamma^0 \) is associated with the \( y \)-axis, and the matrix \( \gamma^5 \) is associated with the \( x \)-axis. The "\( \mathbf{b} \)" vectors are all polar vectors, contrary to the assumption in quantum mechanics that \( \mathbf{b}_0 \) (and only \( \mathbf{b}_0 \)) is an axial vector.

The significance of this difference is that according to classical physics, matter and antimatter are mirror images of each other. [15] This is of course consistent with the fact that "matter to the right is symmetrical with antimatter to the left." [16] With this interpretation of spatial reflection, the mirror image of beta decay of cobalt-60 is simply beta decay of anti-cobalt-60. The mirror image of a left-handed neutrino is a right-handed anti-neutrino. If "parity violating" experiments were regarded as tests of classical versus quantum spatial reflection operators, then the quantum parity operator would have to be rejected.

To obtain a transverse wave solution, we would like to rotate the wave velocity independently of the polarization direction. This can be accomplished using the \( \gamma \) matrices. Multiplying the wave function by \( \exp \left[ -i \gamma^0 (\pi/4) \right] \) rotates the velocity direction by \( \pi/2 \). Since \( \gamma^0 \) corresponds to the \( y \)-axis, the spatial variable \( z \) must be changed to \( x \) in the altered wave function.

The bispinor becomes:

\[
\psi = \sqrt{\omega Q_0} \exp \left[-i (\omega t - k x)/2 \right] \begin{bmatrix}
-i \cos \left(\frac{\omega t - k x}{2}\right) \\
\sin \left(\frac{\omega t - k x}{2}\right) \\
\cos \left(\frac{\omega t - k x}{2}\right) \\
-i \sin \left(\frac{\omega t - k x}{2}\right)
\end{bmatrix}
\]

and the new wave vector potential is \((Q_x, Q_y, Q_z) = (0, 0, Q_0 \sin(\omega t - k x))\).

Other wave variables are:

\[
\begin{align*}
\mathbf{s} &= \partial_t \mathbf{Q} = \frac{1}{2} [\psi^\dagger \mathbf{\sigma} \psi] = (0, 0, \omega Q_0 \cos(\omega t - k x)); \\
\mathbf{c} \nabla \cdot \mathbf{Q} &= -\frac{i}{2} [\psi^\dagger \gamma^5 \psi] = 0; \\
c^2 (\nabla \times \nabla \times \mathbf{Q}) &= -\frac{ic}{2} \left\{ (\nabla \psi)^\dagger \times \gamma^5 \mathbf{\sigma} \psi + \psi^\dagger \gamma^5 \mathbf{\sigma} \times \nabla \psi \right\} \\
&= (0, 0, c^2 k^2 Q_0 \sin(\omega t - k x));
\end{align*}
\]

Arbitrary monochromatic plane waves can be derived from this one by suitable scaling and rotation operations (including appropriate phase changes). Two constants of the motion are:

\[
\text{Re}(\psi^\dagger i \partial_t \psi) = -\text{Re}(\psi^\dagger i c \gamma^5 \mathbf{\sigma} \cdot \nabla \psi) = \omega^2 Q_0.
\]

The "\( \mathbf{b} \)" vectors are:

\[
\begin{align*}
\mathbf{b}_0 &= \psi^\dagger \gamma^0 \mathbf{\sigma} \psi = \mathbf{\hat{y}} \times \omega \mathbf{Q} = (\omega Q_0 \sin(\omega t - k x), 0, 0); \\
\mathbf{b}_4 &= \psi^\dagger \gamma^4 \mathbf{\sigma} \psi = -\omega Q_0 \mathbf{\hat{z}} = (0, 0, -\omega Q_0); \\
\mathbf{b}_5 &= \psi^\dagger \gamma^5 \mathbf{\sigma} \psi = \mathbf{\hat{x}} \times \omega \mathbf{Q} = (0, -\omega Q_0 \sin(\omega t - k x), 0).
\end{align*}
\]

For this transverse plane wave, the current \( \mathbf{b}_5 \) is not aligned with the wave velocity direction. Instead it is proportional to a cross product of the wave vector with the vector potential \( \mathbf{Q} \).

\section*{D. Special Relativity}

The mass term in quantum mechanics involves multiplication of the wave function by \( i \gamma^0 \), which we have shown above is involved in rotation of wave velocity. This fact suggests that particles with mass should be interpreted as
waves whose velocity direction continuously rotates, or as standing waves consisting of a superposition of such waves. This behavior is similar to that of De Broglie waves in a central potential, whose rays follow circular paths between two bounding radii.[17] In the quantum mechanical interpretation of the Dirac equation, the fluctuation of position known as "zitterbewegung" is attributable to the particle undergoing circular motion with diameter equal to the Compton wavelength: \( \lambda_0 = \frac{h}{m_0c} \).[18]

![Figure 1](image)

**FIG. 1.** (a) Model of circular wave propagation with the vertical axis representing the azimuthal direction. (b) Model of helical wave propagation with \( \gamma = 2 \).

The model of particles as circulating or standing waves offers a simple means for understanding special relativity (SR). [19, 20] Model a particle at rest as in Fig. 1(a) by a wave propagating in circles at the speed of light around the \( z \)-axis (horizontal direction) with wave frequency \( f_0 = \frac{m_0c^2}{h} \) and wavelength \( \lambda_0 = \frac{h}{m_0c} \). The gray arrow represents the distance light travels in one unit of time, as measured by a stationary observer. The internal clock ticks once each time the wave traverses the circle. Rotating the wave crests as in Fig. 1(b) to produce helical wave propagation with average velocity \( v \) and relativistic factor \( \gamma = \frac{c}{\sqrt{c^2 - v^2}} \) results in a new wavelength of \( \lambda_0/\gamma \) and frequency \( \gamma f_0 \). The width of the moving wave packet is reduced by a factor of \( 1/\gamma \) (length contraction). Propagation in the azimuthal direction, which measures time, is also reduced by a factor of \( 1/\gamma \) (time dilation). The distance between wave crests along the \( z \)-axis is \( (\lambda_0/\gamma)(c/v) = \frac{h}{(\gamma m_0 v)} = \frac{h}{\gamma p} \), which is the De Broglie wavelength of a moving "particle". Hence the De Broglie wavelength results from a Lorentz boost of a stationary oscillation.

Consider the velocity triangle with hypotenuse \( c \), one side representing average motion \( v \), and a third side \( \sqrt{c^2 - v^2} \) representing circulating motion perpendicular to the average motion. The Pythagorean theorem yields:

\[
e^2 = v^2 + \left( \sqrt{c^2 - v^2} \right)^2.
\]

(63)

Simply multiply each side by \( \gamma m_0 c \), with rest mass \( m_0 \), to obtain the energy-momentum-mass triangle. The Pythagorean theorem now yields:

\[
(\gamma m_0 c)^2 = (\gamma m_0 c v)^2 + (m_0 c^2)^2
\]

(64)

which is equivalent to:

\[
E^2 = (pc)^2 + (m_0 c^2)^2
\]

(65)

This relationship is valid, averaging over the cyclical motion, even if the average motion is in the plane of circulation.[20]

Since the wave equation is Lorentz-invariant and also arises for many different types of waves, SR should be understood as a general property of waves rather than a property of spacetime [19, 20]. A unifying principle of SR, applicable to all waves, is this:

Measurements made by differently moving observers using a particular type of wave are related by Lorentz transformations based on the characteristic wave speed.

Hence the model of the vacuum as an ideal (non-dispersive) elastic solid existing in a Galilean physical spacetime (with wave measurements comprising Minkowski spacetime) is entirely consistent with the laws of SR. The reader
my recall that Maxwell also derived the equations of electromagnetism with the assumption of Galilean spacetime. Curiously, the success of Maxwell’s model sometimes seems to be regarded as evidence that his assumptions were wrong! In the words of Robert Laughlin, “Relativity actually says nothing about the existence or nonexistence of matter pervading the universe, only that any such matter must have relativistic symmetry. It turns out that such matter exists.” [21] Einstein’s postulate of the constancy of the speed of light may be understood as a recognition that all of our measurements are made using waves (including particle-like or standing waves) whose characteristic propagation speed is the speed of light. The current definition of the "meter" guarantees a constant measured speed of light (even though we know that the actual speed of light varies in a gravitational field).

Interpretation of SR as a property of matter rather than spacetime clarifies the analysis of relative motion. Although it is impossible to measure absolute velocity, it is possible to measure absolute acceleration. If an inertial observer detects relativistic changes to accelerated clocks and rulers, it is certain that those changes are real, and they are consistent with the wave nature of matter. Acceleration changes matter, not the spacetime in which the matter moves. Likewise, an accelerated observer should realize that changes seen in external inertial clocks and rulers are not real, but are due to changes in the co-accelerated clocks and rulers used for comparison. Poincaré’s statement that “we have no means of knowing whether it is the magnitude or the instrument that has changed” [22] does not apply to accelerated reference frames.

E. Lagrangian and Hamiltonian

Now we construct a Lagrange density $\mathcal{L}$. Lagrange’s equation of motion for a field variable $\psi$ is

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial [\partial_t \psi]} + \sum_j \frac{\partial}{\partial [\partial_j \psi]} \left( \frac{\partial \mathcal{L}}{\partial \partial_j \psi} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0.$$ (66)

Multiplying Eq. 53 by $i \psi^\dagger$ yields:

$$\psi^\dagger i \partial_t \psi + c \psi^\dagger \gamma^5 \sigma \cdot i \nabla \psi - \psi^\dagger \chi \psi = 0,$$ (67)

Derivatives of $\psi^\dagger$ do not appear in this equation. Therefore we can construct a Lagrangian whose Euler-Lagrange equation has the simple form $\partial \mathcal{L} / \partial \psi^\dagger = 0$:

$$\mathcal{L} = i \psi^\dagger \partial_t \psi + c \psi^\dagger \gamma^5 \sigma \cdot i \nabla \psi - \psi^\dagger \chi \psi.$$ (68)

The imaginary part of the Lagrangian has no physical significance, so we may discard it:[23]

$$\mathcal{L} = \text{Re} \left\{ \psi^\dagger \partial_t \psi + \psi^\dagger c \gamma^5 \sigma \cdot i \nabla \psi - \psi^\dagger \chi \psi \right\}.$$ (69)

From here on the representation of physical quantities as the real part of complex quantities will be implicit. The associated Hamiltonian is:

$$\mathcal{H} = p_x \partial_x \psi - \mathcal{L} = -\psi^\dagger c \gamma^5 \sigma \cdot i \nabla \psi + \psi^\dagger \chi \psi.$$ (70)

If we had kept nonlinear terms in Eq. 25, then the Hamiltonian would contain addition terms including $(1/2) \mathbf{w} \cdot \psi^\dagger (\sigma / 2) \psi$ whose volume integral equals kinetic energy.

The Hamiltonian operator defined by $i \partial_t \psi = H \psi$ is: [1]

$$H \psi = -c \gamma^5 \sigma \cdot i \nabla \psi + \chi \psi.$$ (71)

In quantum mechanics, the Hamiltonian represents energy density. We also saw that for infinitesimal elastic plane waves, the quantities $\text{Re}(\psi^\dagger i \partial_t \psi)$ and $-\text{Re}(\psi^\dagger i c \gamma^5 \sigma \cdot \nabla \psi)$ are equal constants of the motion.

The Hamiltonian is a special case $(T^0_0)$ of the energy-momentum tensor:

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial [\partial_\mu \psi]} \partial_\nu \psi - \mathcal{L} \delta^\mu_\nu.$$ (72)

The conjugate momenta computed from the Lagrangian have the opposite sign of physical quantities. The dynamical (or wave) momentum density $P_i$ is

$$P_i = -T^0_i = -\frac{\partial \mathcal{L}}{\partial [\partial_i \psi]} \partial_t \psi = -\psi^\dagger i \partial_t \psi.$$ (73)
The wave angular momentum density is likewise

\[
\mathbf{L} = -\frac{\partial L}{\partial [\partial_t \psi]} \partial_r \psi = -i\psi^\dagger \partial_r \psi = -i\psi^\dagger \frac{\partial}{\partial r} \partial_r \psi
\]

\[
= -\mathbf{r} \times \psi^\dagger \nabla \psi = \mathbf{r} \times \mathbf{P}.
\]

These dynamical variables are consistent with those of quantum mechanics. For total momentum \((\mathbf{P} + \mathbf{p})\) and angular momentum \((\mathbf{L} + \mathbf{s})\), we must combine the wave and medium contributions:

\[
\mathbf{P} + \mathbf{p} = -\psi^\dagger i\nabla \psi + \frac{1}{2} \nabla \times \psi^\dagger \sigma \psi;
\]

\[
\mathbf{L} + \mathbf{s} = -\mathbf{r} \times \psi^\dagger i\nabla \psi + \psi^\dagger \sigma \psi.
\]

The angular momentum operator is equivalent to that of quantum mechanics. The addition of intrinsic momentum to the wave momentum makes the energy-momentum tensor symmetric, as required for general relativity [24–26].

If the wave function is an eigenfunction of the spin component \(s_z\) with total spin \(\hbar/2\), then the wave function should be normalized to \(\int_V \psi^\dagger \sigma \psi dV = \hbar\). However, it is customary to normalize the wave function to unity, so all operators should be modified to include a factor of \(\hbar\):

\[
H\psi = -c\gamma^5 \sigma \cdot i\hbar \nabla \psi + \hbar \chi \psi
\]

\[
\mathbf{P} + \mathbf{p} = -\psi^\dagger i\hbar \nabla \psi + \frac{1}{2} \nabla \times \psi^\dagger \sigma \hbar \psi
\]

\[
\mathbf{L} + \mathbf{s} = -\mathbf{r} \times \psi^\dagger i\hbar \nabla \psi + \psi^\dagger \sigma \hbar \psi.
\]

V. WAVE INTERACTIONS

Suppose we have two Dirac wave functions \(\psi_A\) and \(\psi_B\), representing particle-like waves \(A\) and \(B\). Adding the wave functions yields a total wave function \(\psi_T\) satisfying:

\[
\psi_T^\dagger \sigma \psi_T = (\psi_A + \psi_B)^\dagger \sigma (\psi_A + \psi_B)
\]

\[
= \psi_A^\dagger \sigma \psi_A + \psi_B^\dagger \sigma \psi_B
\]

\[
+ \psi_A^\dagger \sigma \psi_B + \psi_B^\dagger \sigma \psi_A.
\]

(80)

Since the spins must be additive, the total wave function is not generally the sum of the individual wave functions. However, we can treat the wave functions as being independent if the interference terms cancel [1]. This cancelation imposes a vector constraint on the wave functions:

\[
\psi_A^\dagger \sigma \psi_B + \psi_B^\dagger \sigma \psi_A = 0.
\]

(81)

Assuming either of the waves to be a spin eigenfunction everywhere, one component of this constraint requires the wave functions to anti-commute:

\[
\psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A = 0.
\]

(82)

For waves representing identical particles, this is the Pauli exclusion principle. Hence we can conclude that standing waves described by spin eigenfunctions are fermions.

The anti-commutation of wave functions is not true in general, but we can force the cancellation by introducing a phase shift at each point between the two wave functions. Such phase shifts have no effect on the actual dynamics of the total wave, but allow us to pretend that each particle wave maintains its separate identity even though there is actually only one combined wave. Of course, this procedure is only valid if the particles interact weakly enough to remain distinguishable during the interaction. This limitation does not invalidate the basic premise that physical quantities are fully determined by the spin density field.

The phase shift \((\delta)\) is determined from the constraint:

\[
\Re(\psi_A^\dagger \exp(i\delta)\psi_B) = 0,
\]

(83)
or

\[ \text{Re}(\psi_A^\dagger \psi_{B^r}) \cos \delta - \text{Im}(\psi_A^\dagger \psi_{B^r}) \sin \delta = 0. \]  

(84)

This yields:

\[ \tan \delta = \frac{\text{Re}(\psi_A^\dagger \psi_{B^r})}{\text{Im}(\psi_A^\dagger \psi_{B^r})}. \]  

(85)

If we let \( \psi_A^\dagger \psi_{B^r} = g \exp i\beta \), then:

\[ \tan \delta = \cot \beta. \]  

(86)

Therefore the phase angles are related by:

\[ \delta = \frac{\pi}{2} - \beta \pm n\pi \]  

(87)

where \( n \) is an integer. Note that \( \delta \) is only unique within an arbitrary multiple of \( \pi \).

For example, if \( \psi_A \) and \( \psi_B \) are real-valued except for overall phase factors \( \exp(i\varphi_A) \) and \( \exp(i\varphi_B) \), then the interaction phase factor is:

\[ \delta = \frac{\pi}{2} + \varphi_A - \varphi_B \pm n\pi \]  

(88)

The overall phase shift at each point must be divided between the two "particles" so that they experience equal and opposite forces according to Newton's third law.

Suppose we start with two wave functions \( \psi_A \) and \( \psi_B \), initially non-overlapping and normalized to one. As they approach each other, the total wave function is \( \psi_T = \exp[-i\delta_A] \psi_A + \exp[-i\delta_B] \psi_B \), with the phase shifts satisfying \( \delta_A - \delta_B + \pi/2 = \delta \) in order to cancel the interference terms.

For example, if the wave functions have separable phase factors of \( \exp[i\alpha_A] \) and \( \exp[i\alpha_B] \), then a simple prescription would be

\[ \begin{align*}
\delta_A &= f(r,t)\alpha_B - (1 - g(r,t))\alpha_A, \\
\delta_B &= g(r,t)\alpha_A - (1 - f(r,t))\alpha_B,
\end{align*} \]  

(89)

where the fractions \( f(r,t) \) and \( g(r,t) \) are chosen to make the force density on the two particles equal and opposite. However, the wave functions generally may not have such simply separable phase factors.

The phase-shifted wave functions are independent, so they satisfy the free-particle wave equation, e.g.

\[ i\hbar \partial_t \left( \exp[-i\delta_A] \psi_A \right) = H_0 \exp[-i\delta_A] \psi_A. \]  

(90)

We take \( \psi_A \) to be an electron wave function with free particle hamiltonian \( H_0 \psi_A = (-c\gamma^5 \sigma \cdot \hbar \nabla - im_c c^2 \gamma^0) \psi_A \).

Expanding the Dirac equation for \( \psi_A \) yields:

\[ (-i\hbar \partial_t + \hbar \partial_A) \psi_A = (-c\gamma^5 \sigma \cdot (i\hbar \nabla \delta_A + \hbar \nabla) - im_c c^2 \gamma^0) \psi_A \]  

(91)

The modified Hamiltonian is:

\[ H' \psi = -i\hbar \partial_t \psi = \hbar(\partial_t \delta_A) \psi + c\gamma^5 \sigma \cdot i\hbar(\nabla - \nabla \delta_A) \psi + im_c c^2 \gamma^0 \psi \]  

(92)

The time derivative of wave momentum density (still using the momentum operator \( P = -i\hbar \nabla \)) is:

\[ \begin{align*}
d_t P &= \psi^\dagger \exp[i\delta_A][H', -i\hbar \nabla] \exp[-i\delta_A] \psi \\
&= -\psi^\dagger \left[ \hbar \nabla (\partial_t \delta_A) - \hbar \nabla (\gamma^5 \sigma \cdot \nabla \delta_A) + \hbar c (\gamma^5 \sigma \cdot \nabla) \nabla \delta_A \right] \psi \\
&= \psi^\dagger \left[ -\hbar \nabla (\partial_t \delta_A) + \hbar c \gamma^5 \sigma \times (\nabla \times (\nabla \delta_A)) \right] \psi
\end{align*} \]  

(93)

We can obtain the Lorentz force on the negatively charged electron by identifying the electric and magnetic fields as:

\[ -eB = \nabla \times (\hbar \nabla \delta_A) \]  

\[ -eE = -\nabla (\hbar \partial_t \delta_A) \]  

(94)
If we define $A = -(\hbar/e)\nabla \delta_A$ then:

$$
\begin{align*}
B &= \nabla \times A \\
E &= -\partial_t A.
\end{align*}
$$

(95)

Others have similarly identified the vector potential $A$ as the gradient of a multivalued field.[27–29] This interpretation is also consistent with Synge’s ”primitive quantization” in which Planck’s constant $\hbar$ represents the action for a single wave cycle. [30]

The lack of a scalar potential defines a Weyl (or temporal) electromagnetic gauge. We can perform a Helmholtz decomposition of $E$ to obtain:

$$
E = -\nabla \Phi - \partial_t A'
$$

(96)

where $A'$ is the divergence-free part of $A$.

Suppose the field $\psi_B$ produces a well-defined phase shift on $\psi_A$ that is unique within an offset of some multiple of $\pi$:

$$
\oint \nabla \delta_A \cdot d\ell = n\pi
$$

(97)

for some integer $n$. Stoke’s law yields quantization of magnetic flux:

$$
\int \int B \cdot \hat{n} dS = \oint A \cdot d\ell = \pm n\pi \frac{\hbar}{e} = \pm n\frac{\hbar}{2e}.
$$

(98)

A. Electron Interactions

Alternatively, suppose that the interaction phase shift has the form:

$$
\delta_A = e^2(m_\phi \phi - \omega t)/(4\pi e_0 \hbar \omega r),
$$

(99)

with $\hbar = m_e c^2$.

The vector potential is then:

$$
A = -(\hbar/e)\nabla \delta_A = - \left[ (-\frac{e}{4\pi e_0 \omega r^2})(m_\phi \phi - \omega t)\hat{r} + \frac{me}{4\pi e_0 \omega r^2 \sin \theta} \hat{\phi} \right]
$$

(100)

The electric field is that of a negative point charge:

$$
E = -\partial_t A = -\frac{e}{4\pi e_0 r^2} \hat{r}.
$$

(101)

The magnetic flux in the plane $z = 0$ is:

$$
\oint A \cdot d\ell = \frac{m_\phi e}{2\epsilon_0 \omega r} = \frac{m_\phi \mu_0 \hbar e}{2m_e r}.
$$

(102)

For $m_\phi = 1/2$, this is $\mu_0 \hbar e/(4m_e r)$. For comparison, the electron spin dipole moment is $M \approx e\hbar/(2m_e r)$ to first order, and its magnetic flux in the plane $z = 0$ is:

$$
\int \int B \cdot \hat{n} dS = \int -2\pi \frac{\mu_0 M}{4\pi r^3} r dr = \frac{\mu_0 M}{2r} = \mu_0 \hbar e/(4m_e r),
$$

(103)

in agreement with our calculation. Hence this choice of phase shift yields the correct relationship between the electron’s electric field and magnetic flux. However, it is not clear what electron wave function would give rise to this phase shift.
B. Maxwell’s Equations

The electromagnetic fields defined above are also subject to Maxwell’s equations. The definitions of \( \mathbf{E} \) and \( \mathbf{B} \) imply Faraday’s Law and Gauss’ magnetic law:

\[
\nabla \times \mathbf{E} = -\partial_t \mathbf{B},
\]
\[
\nabla \cdot \mathbf{B} = 0.
\]

Gauss’ electric law and Ampere’s law define the charge and current densities (\( \rho_c \) and \( \mathbf{J} \), respectively):

\[
\nabla \cdot \mathbf{E} = \frac{\hbar c}{e} (\nabla^2 \partial_t A) \equiv \rho_c / \varepsilon_0
\]
\[
\nabla \times \mathbf{B} - \frac{1}{c^2} \partial_t \mathbf{E} = \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \partial_t^2 \mathbf{A}
\]
\[=
\nabla \times (\nabla \times (-\frac{\hbar c}{e} \nabla \delta_A)) + \frac{\hbar}{ec} (\nabla \partial_t^2 \delta_A)
\]
\[= \mu_0 \mathbf{J}
\]

These definitions of charge and current densities are consistent with the continuity equation:

\[
\partial_t \rho_c + \nabla \cdot \mathbf{J} = 0.
\]

Hence particle-like waves in an elastic solid behave like fermions, with electromagnetic potentials derived from phase shifts that result from wave interference.

C. Quantum Electrodynamics

It is customary in quantum mechanics textbooks to replace \( \psi^\dagger \) with \( \psi^\dagger \gamma^0 \gamma^0 \) and define \( \tilde{\psi} \equiv \psi^\dagger \gamma^0 \) and the "4-vector" of matrices \( \gamma^\mu \equiv (\gamma^0, \gamma^0 \sigma) \). The 4-potential is \( A_\mu = (\Phi, -\mathbf{A}) \) and the 4-current (\( \rho, -\mathbf{J} \)) is \( J^\mu = q \bar{\psi} \gamma^\mu \psi \). These changes of variables are intended to make the theory look more "relativistic". It is also common to use "natural" units with \( \mu_0 = e_0 = c = 1 \). Using this notation, the Lagrangian for two interacting electrons is:

\[
\mathcal{L} = \bar{\psi}_A e^{-i \lambda^\mu [\gamma^\mu \partial_\mu - m_A]} e^{i \lambda^\beta \bar{\psi}_B} + \bar{\psi}_B e^{-i \lambda^\mu [\gamma^\mu \partial_\mu - m_B]} e^{i \lambda^\beta \bar{\psi}_B}
\]

We have seen that derivatives of the interaction phase shifts yield the electromagnetic potentials. Now we assume that all phase factors have a similar interpretation but with 4-current modified by a factor of two so that \( J^\mu = 2q \bar{\psi} \gamma^\mu \psi \). In other words, a charged particle interacts with its own potentials:

\[
q \bar{\psi} \gamma^\mu \partial_\mu \psi = -\frac{1}{2} J^\mu A_\mu.
\]

Treating particle \( B \) as an external field with the interaction phase shift incorporated into the wave function so that it is independent of wave \( A \), we have:

\[
\mathcal{L} = \bar{\psi}_A e^{-i \lambda^\mu [\gamma^\mu \partial_\mu - m_A]} e^{i \lambda^\beta \bar{\psi}_B} - \frac{1}{2} J^\mu A_\mu - \bar{\psi}_B m_B \psi_B
\]

Interpreting the potentials in terms of a phase \( \alpha_B \), the contribution from particle \( B \) is:

\[
-\frac{1}{2} J^\mu A_\mu = \frac{1}{2} \rho_c \partial_t \alpha_B + \frac{1}{2} \mathbf{J} \cdot \nabla \alpha_B - \bar{\psi}_B m_B \psi_B
\]

Relationships between potentials and currents are given in Eq. 105. According to Green’s first identity:

\[
\int_V (\partial_t \alpha_B) \nabla^2 (\partial_t \alpha_B) dV = -\int (\nabla (\partial_t \alpha_B))^2 dV + \int_{\partial V} (\partial_t \alpha_B) \mathbf{n} \cdot \nabla (\partial_t \alpha_B) dS
\]

Similarly:

\[
\int_V (\nabla \times \nabla \times \mathbf{A}) \cdot dV = \int_V (\nabla \times \mathbf{A})^2 dV - \int_{\partial V} \mathbf{A} \times (\nabla \times \mathbf{A}) dS
\]
Neglecting boundary integrals yields:

$$-\frac{1}{2} \int_V J^\mu A_\mu dV = -\frac{1}{2} \int_V (E^2 - B^2) dV + \int_V (\partial^2 \nabla \alpha B) \cdot \nabla \alpha B dV$$  \hspace{1cm} (113)

If we substitute this last equality into the Lagrange density, we obtain:

$$L = \bar{\psi} A e^{-i\Delta [\gamma^\mu \partial_\mu - m_A]} e^{i\Delta \psi} \psi - \frac{1}{2} E^2 - B^2 + (\partial^2 \nabla \alpha B) \cdot \nabla \alpha B - \bar{\psi} B m B \psi$$  \hspace{1cm} (114)

In terms of the electromagnetic tensor $F^{\mu\nu}$:

$$L = \bar{\psi} A e^{-i\Delta [\gamma^\mu \partial_\mu - m_A]} e^{i\Delta \psi} \psi - \frac{1}{4} F^{\mu\nu} F^{\rho\sigma} + (\partial^2 \nabla \alpha B) \cdot \nabla \alpha B - \bar{\psi} B m B \psi$$  \hspace{1cm} (115)

If we ignore the last two terms, this is the un-quantized Lagrange density of quantum electrodynamics. The proposed electron phase shift in Eq. 99 would have $\partial^2 \alpha B \propto \omega \propto m_e$, so it is plausible that the last two terms would cancel. Of course we have also neglected nonlinear terms in the wave equation. External currents are typically added in the form $J^\mu A_\mu$ with $J^\mu$ treated as independent of $A_\mu$ for the purpose of variations. However, we would add $(1/2)J^\mu A_\mu$ with the understanding that $J^\mu$ is a function of $A_\mu$ so that the variation with respect to $A_\mu$ yields $J^\mu$ in a manner similar to Eq. 15.

Although speculative, we have outlined a possible path from the classical model of an elastic solid to quantum electrodynamics (QED). The interpretation of QED, and by extension the standard model, is that it represents a decomposition of the spin density field into interacting elementary particles.

VI. DISCUSSION

We have demonstrated that the Dirac equation may be derived from a description of infinitesimal wave motion in an elastic solid. Unlike the non-relativistic Schrödinger equation, the Dirac equation is fully relativistic and physically realistic. Each of the variables, including spin angular momentum, has a clear physical interpretation.

Thomas Jefferson famously wrote that “Ignorance is preferable to error; and he is less remote from the truth who believes nothing, than he who believes what is wrong” [31]. The non-relativistic Schrödinger equation is obviously wrong, and is therefore a poor choice for introducing students to the wave nature of matter. Students should first be taught the physical basis for the Dirac equation, after which the Schrödinger equation may be derived as an approximation in order to simplify the mathematics.

With finite motion, nonlinear terms would be added to the linear wave equation. Nonlinearity is a possible reason for quantized amplitudes. Many researchers have attempted to quantize the Dirac equation by adding nonlinear terms. [32–39]

The model of the vacuum as an elastic solid also offers a good introduction to general relativity. Gravity, at least when weak, may be interpreted as ordinary refraction of waves toward regions whose wave speed is decreased by the presence of energy. [40–42] The density of an elastic solid may also be increased by stress-induced compression. For example, twisting a rubber band induces a tension that tends to shorten it.

We showed how a model of stationary matter as standing waves gives rise to the De Broglie wavelength for moving particles, and gave a plausible explanation of magnetic flux quantization. Recent research has revealed that classical physical systems can reproduce other quantum phenomena as well. In particular, silicone droplets bouncing on a vibrating water tank can exhibit single-particle diffraction and interference, wave-like probability distributions, tunneling, quantized orbits, and orbital level splitting.[43–48] Students (and their professors) should be aware that many quantum behaviors have classical analogues.

VII. CONCLUSIONS

This paper offers a new approach for introducing students to the wave nature of matter, based on a classical wave description of infinitesimal, incompressible motion in an elastic solid. Unlike the non-relativistic Schrödinger equation, this approach to wave mechanics is fully relativistic. Spin angular momentum is the field whose curl is equal to twice the incompressible momentum density. The second-order wave equation is transformed into a first-order Dirac equation, and sample plane wave solutions are given. A Lagrangian and Hamiltonian are constructed, from which the dynamical operators of relativistic quantum mechanics are derived. A model of stationary matter as circulating waves yields the relativistic energy-momentum equation for relativistic particles. Wave interference gives rise to the
Pauli exclusion principle and electromagnetic potentials, with plausible explanations for magnetic flux quantization and electric point charges. Hence classical wave theory can be a powerful tool for mathematical modeling of the wave properties of matter.